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JHEP05(2009)075

(http://iopscience.iop.org/1126-6708/2009/05/075)

Published by IOP Publishing for SISSA
Received: *March 27, 2009*
Accepted: *May 11, 2009*
Published: *May 18, 2009*

Family gauge symmetry as an origin of Koide's mass formula and charged lepton spectrum

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Abstract: Koide's mass formula is an empirical relation among the charged lepton masses which holds with a striking precision. We present a model of charged lepton sector within an effective field theory with $\mathrm{U}(3) \times \mathrm{SU}(2)$ family gauge symmetry, which predicts Koide's formula within the present experimental accuracy. Radiative corrections as well as other corrections to Koide's mass formula have been taken into account. We adopt a known mechanism, through which the charged lepton spectrum is determined by the vacuum expectation value of a 9-component scalar field Φ. On the basis of this mechanism, we implement the following mechanisms into our model: (1) The radiative correction induced by family gauge interaction cancels the QED radiative correction to Koide's mass formula, assuming a scenario in which the U(3) family gauge symmetry and $\mathrm{SU}(2)_L$ weak gauge symmetry are unified at 10^2–10^3 TeV scale; (2) A simple potential of Φ invariant under $\mathrm{U}(3) \times \mathrm{SU}(2)$ leads to a realistic charged lepton spectrum, consistent with the experimental values, assuming that Koide's formula is protected; (3) Koide's formula is stabilized by embedding $\mathrm{U}(3) \times \mathrm{SU}(2)$ symmetry in a larger symmetry group. Formally fine tuning of parameters in the model is circumvented (apart from two exceptions) by appropriately connecting the charged lepton spectrum to the boundary (initial) conditions of the model at the cut-off scale. We also disucss some phenomenological implications.

Keywords: Beyond Standard Model, Quark Masses and SM Parameters, Gauge Symmetry

doi:10.1088/1126-6708/2009/05/075

Contents

1 Introduction

Among various properties of elementary particles, the spectra of the quarks and leptons exhibit unique patterns, and their origin still remains as a profound mystery. Within the Standard Model (SM) of elementary particles, the origin of the masses and mixings of the quarks and leptons is attributed to their interactions with the (as yet hypothetical) Higgs boson. Namely, these are the Yukawa interaction in the case of the charged leptons and quarks, and possibly the interaction represented by dimension-5 operators in the case of the left-handed neutrinos. Even if these interactions will be confirmed experimentally in the future, since the couping constants of these interactions are free parameters of the theory, the underlying mechanism how the texture of these couplings is determined would remain unrevealed.

There have been many attempts to approach the mystery of the fermion masses by identifying empirical relations among the observed fermion masses and exploring underlying physics that would lead to such relations. In particular, Koide's mass formula is an empirical relation among the charged lepton masses given by [1]

$$\frac{\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau}}{\sqrt{m_e+m_\mu+m_\tau}}=\sqrt{\frac{3}{2}}\,, \tag{1.1}$$

which holds with a striking precision. In fact, substituting the present experimental values of the charged lepton masses [2], the formula is valid within the present experimental accuracies. The relative experimental error of the left-hand side (l.h.s.) of eq. (1.1) is dominated by $\frac{1}{\sqrt{6}}(m_\mu/m_\tau)^{1/2}(\Delta m_\tau/m_\tau)$ (Δm_τ is the experimental error of m_τ) and is of order 10^{-5}. A simple mnemonic of the relation (1.1) is that the angle between the two vectors $(\sqrt{m_e},\sqrt{m_\mu},\sqrt{m_\tau})$ and $(1,1,1)$ equals $45°$ [3].

Given the remarkable accuracy with which Koide's mass formula holds, many speculations have been raised as to existence of some physical origin behind this mass formula [3–9]. Despite the attempts to find its origin, so far no realistic model or mechanism has been found which predicts Koide's mass formula within the required accuracy. The most serious problem one faces in finding a realistic model or mechanism is caused by the QED radiative correction [7]. Even if one postulates some mechanism at a high energy scale that leads to this mass relation, the charged lepton masses receive the 1-loop QED radiative corrections given by

$$m_i^{\rm pole}=\left[1+\frac{\alpha}{\pi}\left\{\frac{3}{4}\log\left(\frac{\mu^2}{\bar{m}_i(\mu)^2}\right)+1\right\}\right]\bar{m}_i(\mu)\,. \tag{1.2}$$

$\bar{m}(\mu)$ and $m^{\rm pole}$ denote the running mass defined in the modified-minimal-subtraction scheme ($\overline{\rm MS}$ scheme) and the pole mass, respectively; μ represents the renormalization scale. It is the pole mass that is measured in experiments. Suppose $\bar{m}_i(\mu)$ (or the corresponding Yukawa couplings $\bar{y}_i(\mu)$) satisfy the relation (1.1) at a high energy scale $\mu\gg M_W$. Then $m_i^{\rm pole}$ do not satisfy the same relation [6, 7]: eq. (1.1) is corrected by approximately 0.1%, which is 120 times larger than the present experimental error. Note that this correction originates only from the term $-3\alpha/(4\pi)\times\bar{m}_i\log(\bar{m}_i^2)$ of eq. (1.2), since the other terms, which are of the form const. $\times\,\bar{m}_i$, do not affect the relation (1.1). This is because, the latter corrections only change the length of the vector $(\sqrt{m_e},\sqrt{m_\mu},\sqrt{m_\tau})$ but not the direction. We also note that $\log(\bar{m}_i^2)$ results from the fact that $\bar{m}_i$ plays a role of an infrared (IR) cut-off in the loop integral.

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The 1-loop weak correction is of the form const. $\times \bar{m}_i$ in the leading order of $\bar{m}_i^2/M_W^2$ expansion; the leading non-trivial correction is $\mathcal{O}(G_F\bar{m}_i^3/\pi)$ whose effect is smaller than the current experimental accuracy. Other radiative corrections within the SM (due to Higgs and would-be Nambu-Goldstone bosons) are also negligible.

Thus, if there is indeed a physical origin to Koide's mass formula at a high energy scale, we need to account for a correction to the relation (1.1) that cancels the QED correction. Since such a correction is absent up to the scale of $\mathcal{O}(M_W)$ to our present knowledge, it must originate from a higher scale. Then, there is a difficulty in explaining why the size of such a correction should coincide accurately with the size of the QED correction which arises from much lower scales. There are also other less serious, but important questions that are often asked: (1) Why do not quark masses satisfy the same or a similar relation? (2) In Koide's formula the three lepton masses appear symmetrically. Then why is there a hierarchy among these masses, $m_e \ll m_\mu \ll m_\tau$? If there is indeed a physical origin to Koide's mass formula, there must be reasonable answers to all of these questions.

Among various existing models which attempt to explain origins of Koide's mass formula, we find a class of models particularly attractive [5, 10, 13]. These are the models which predict the mass matrix of the charged leptons to be proportional to the square of the vacuum expectation value (VEV) of a 9-component scalar field (we denote it as Φ) written in a 3-by-3 matrix form:

$$\mathcal{M}_\ell \propto \langle\Phi\rangle\langle\Phi\rangle \,. \tag{1.3}$$

Thus, $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ is proportional to the diagonal elements of $\langle\Phi\rangle$ in the basis where it is diagonal. The VEV $\langle\Phi\rangle$ is determined by minimizing the potential of scalar fields in each model. Hence, the origin of Koide's formula is attributed to the specific form of the potential which realizes this relation in the vacuum configuration. Up to now, no model is complete with respect to symmetry: Every model requires either absence or strong suppression of some of the terms in the potential (which are allowed by the symmetry of that model), without justification.

In this paper, we study possible connections between family (horizontal) gauge symmetries and Koide's formula and the charged lepton spectrum. These will be discussed within the context of an effective field theory (EFT) which is valid below some cut-off scale. In particular we address the following points:

(i) We propose a possible mechanism for cancellation of the QED radiative correction to Koide's mass formula.

(ii) We propose a mechanism that produces the charged lepton spectrum, which is hierarchical and approximates the experimental values, under the assumption that Koide's formula is protected by some other mechanism.

(iii) We present a model of charged lepton sector based on U(3) $\times$ SU(2) family gauge symmetry, incorporating the mechanisms (i)(ii). A new mechanism that stabilizes Koide's formula is incorporated in this model.

(Among these, we have reported the main point of (i) separately in [11].)

In our study we adopt the mechanism eq. (1.3) for generating the charged lepton masses at tree level of EFT, for the following reasons. First, the mechanism allows for transparent and concise perturbative analyses of models, which is crucial in keeping radiative corrections under control. This may be contrasted with models with other mass generation mechanisms, such as dynamical symmetry breaking or composite lepton models, which typically involve strong interactions. Secondly, since Φ is renormalized multiplicatively, the structure of radiative corrections becomes simple, as opposed to cases in which VEVs of more than one scalar fields contribute to the charged lepton spectrum. In short, this type of mass generation mechanism is pertinent to serious analyses of radiative corrections to Koide's formula, which is a distinguished aspect of this study.

We alert in advance that we do not solve the hierarchy problem or fine tuning problem of the electroweak scale. We cannot explain how to stabilize the electroweak symmetry-breaking scale against other higher scales included in our model. Solution to this problem is beyond the scope of this paper.

The paper is organized as follows. In section 2, we explain philosophy of our analysis using EFT and argue for its validity and usefulness. We also give a brief overview of the ideas presented in this paper. In section 3, we explain the mechanism for cancelling the QED corrections to Koide's formula. In section 4, we present a potential for generating a realistic charged lepton spectrum, assuming that Koide's formula is protected. In section 5, we analyze a minimal potential whose vacuum corresponds to a desired lepton spectrum. In section 6, we extend the potential by including another field. In section 7, we introduce a higher-dimensional operator which generates the lepton masses and compute corrections to Koide's formula. In section 8, we discuss the energy scales and unsolved questions in our model. In section 9, we discuss phenomenological implications of our model. In section 10 summary and discussion are given. Technical details are collected in appendices.

2 EFT approach and brief overview of the model

Throughout this paper, we consider an EFT which is valid up to some cut-off scale denoted by Λ ($\gg M_W$). In this EFT, we assume that the charged lepton masses are induced by a higher-dimensional operator

$$\mathcal{O} = \frac{\kappa(\mu)}{\Lambda^2}\, \bar{\psi}_{Li}\, \Phi_{ik}\, \Phi_{kj}\, \varphi\, e_{Rj} \tag{2.1}$$

(or by other similar operators, as will be described later). Here, $\psi_{Li} = (\nu_{Li}, e_{Li})^T$ denotes the left-handed lepton $\mathrm{SU}(2)_L$ doublet of the i-th generation; e_{Rj} denotes the right-handed charged lepton of the j-th generation; φ denotes the Higgs doublet field. They are respectively assigned to the standard representations of the SM gauge group. By contrast, a 9-component scalar field Φ is absent in the SM and a singlet under the SM gauge group. We suppressed all the indices except for the generation (family) indices $i, j, k = 1, 2, 3$. (Summation over repeated indices is understood throughout the paper unless otherwise stated.) The dimensionless Wilson coefficient of this operator is denoted as $\kappa(\mu)$. Once Φ acquires a VEV, the operator $\mathcal{O}$ will effectively be rendered to the Yukawa interactions of the SM; after the Higgs field also acquires a VEV, $\langle\varphi\rangle = (0, v_{\rm ew}/\sqrt{2})^T$ with $v_{\rm ew} \approx 250\,\mathrm{GeV}$, the operator will induce the charged-lepton mass matrix of the form eq. (1.3) at tree level:

$$\mathcal{M}_\ell^{\rm tree} = \frac{\kappa\, v_{\rm ew}}{\sqrt{2}\Lambda^2}\langle\Phi\rangle\langle\Phi\rangle\,. \tag{2.2}$$

For a moment, let us assume that the dimension-4 Yukawa interactions $y_{ij}\,\bar{\psi}_{Li}\varphi e_{Rj}$ are prohibited by some mechanism. This will be imposed explicitly by a symmetry in our model to be discussed through sections 3-9.

We now explain philosophy of our analysis using EFT. Conventionally a more standard approach for explaining Koide's mass formula has been to construct models within renormalizable theories. Nevertheless, the long history since the discovery of Koide's formula shows that it is quite difficult to construct a viable renormalizable model for explaining Koide's relation. It is likely that we are missing some essential hints to achieve this goal, if the relation is not a sheer coincidence. In this paper we will show that, within EFT, explanation of Koide's formula is possible by largely avoiding fine tuning of parameters. Consistency conditions (with respect to symmetries of the theory) can be satisfied relatively easily in EFT, or in other words, they can be replaced by reasonable boundary conditions of EFT at the cut-off scale Λ without conflicting symmetry requirements of the theory. (See section 5.) Even under this less restrictive theoretical constraints, we may learn some important hints concerning the relation between the lepton spectrum and family symmetries. These are the role of specific family gauge symmetry in canceling the QED correction, the role of family symmetry in stabilizing Koide's mass relation, or the role of family symmetry in realizing a realistic charged lepton spectrum consistently with experimental values. These properties do not come about separately but are closely tied with each other. These features do not seem to depend on details of more fundamental theory above the cut-off scale Λ but rather on some general aspects of family symmetries and their breaking patterns. Thus, we consider that our approach based on EFT would be useful even in the case in which physics above the scale Λ is obscure and may involve some totally unexpected ingredients —— as it was the case with chiral perturbation theory before the discovery of QCD.

Before discussing radiative corrections within EFT, one would be worried about effects of higher-dimensional operators suppressed in higher powers of $1/\Lambda$. Indeed, using the values of tau mass and the electroweak symmetry breaking scale $v_{\rm ew}$, one readily finds that $v_3/\Lambda \gtrsim 0.1$ (v_i are the diagonal elements of $\langle\Phi\rangle$ in the basis where it is diagonal). Hence, naive dimensional analysis indicates that there would be corrections to Koide's formula of order 10% even at tree level. We now argue that this is not necessarily the case within the scenario under consideration. We may divide the corrections into two parts. These are (i) $1/\Lambda^n$ corrections to the operator $\mathcal{O}$ of eq. (2.1) (the operator which reduces to the SM Yukawa interactions after Φ is replaced by its VEV), and (ii) $1/\Lambda^n$ corrections to the VEV of Φ.

Concerning the corrections (i), we may consider the following example.[1] Suppose that

[1] A similar mechanism is used in [10, 12] to induce $\mathcal{O}$; corrections by higher-order terms in $1/\Lambda^n$ have not been discussed, however.

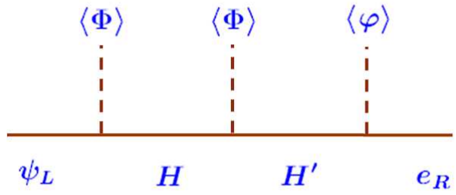

Figure 1. Diagram which induces the higher-dimensional operator $\mathcal{O} = \frac{\kappa(\mu)}{\Lambda^2}\,\bar{\psi}_{Li}\,\Phi_{ik}\,\Phi_{kj}\,\varphi\,e_{Rj}$.

the operator $\mathcal{O}$ is induced from the interactions

$$\mathcal{L} = y_1\,\bar{\psi}_{Li}\Phi_{ij}H_{Rj} + M\,\bar{H}_{Ri}H_{Li} + y_2\,\bar{H}_{Li}\Phi_{ij}H'_{Rj} + M'\bar{H}'_{Ri}H'_{Li} + y_3\,\bar{H}'_{Li}\varphi e_{Ri} + (\text{h.c.}) \tag{2.3}$$

through the diagram shown in figure 1, after fermions $H_{L,R}$ and $H'_{L,R}$ have been integrated out. Fermions $H_{L,R}$ and $H'_{L,R}$ are assigned to appropriate representations of the SM gauge group such that the above interactions become gauge singlet. For instance, in the case that $v_3/M' \gtrsim 3$, $y_1, y_2, y_3 \approx 1$ and $v_{\rm ew}/M' < 3\times10^{-3}$, one finds, by computing the mass eigenvalues,[2] that the largest correction to the lepton spectrum eq. (2.2) arises from the operator $-\dfrac{y_1^3y_2^3y_3}{2M^3M'^3}\,\bar{\psi}_L\Phi^6\varphi e_R$; its contribution to the tau mass is $\delta m_\tau/m_\tau = (m_\tau/v_{\rm ew})^2 \approx 5\times10^{-5}$. This translates to a correction to Koide's relation of 3×10^{-6}, due to the suppression factor $\frac{1}{\sqrt{6}}\,(m_\mu/m_\tau)^{1/2}(\delta m_\tau/m_\tau)$. Thus, this is an example of underlying mechanism that generates the operator $\mathcal{O}$ without generating higher-dimensional operators conflicting the current experimental bound. If we introduce even more (non-SM) fermions to generate the leading-order operator $\mathcal{O}$, one can always find a pattern of spectrum of these fermions, for which higher-dimensional operators are sufficiently suppressed, since the number of adjustable parameters increases. (Another example of underlying mechanism may be the one proposed in [8], based on the idea of [14].) In general, sizes of higher-dimensional operators depend heavily on underlying dynamics above the cut-off scale.

Let us restrict ourselves within EFT. If we introduce only the operator $\mathcal{O}$, by definition this is the only contribution to the charged lepton spectrum at tree level. Whether loop diagrams induce higher-dimensional operators which violate Koide's relation is an important question, and a detailed analysis is necessary. This is the subject of the present study, where the result depends on the mechanisms how Koide's formula is satisfied and how the charged lepton spectrum is determined, even within EFT. The conclusion is as follows. Within the model to be discussed in sections 3–8, the class of 1-loop diagrams shown in figure 2 do not generate operators that violate Koide's relation sizably; see section 7. (There is another type of 1-loop diagrams that possibly cancels the QED correction; see section 3.) In fact, we do not find any loop-induced higher-dimensional operators, which violate Koide's relation in conflict with the current experimental bound.

Concerning the corrections (ii), in our analysis we introduce specific family gauge symmetries and their breaking patterns such that the corrections (ii) are suppressed.

[2] Since the values of m_τ and $v_{\rm ew}$ are known, once we choose the values of $v_3/M'(\gtrsim 3)$ and $y_1, y_2, y_3(\approx 1)$, the value of $v_3/M(\lesssim 0.03)$ will be fixed. Then the mass eigenvalues corresponding to the SM charged leptons can be computed in series expansion in the small parameters $v_{\rm ew}/M'$, v_i/M and $v_i^2/(MM') = \sqrt{2}m_i/v_{\rm ew}$.

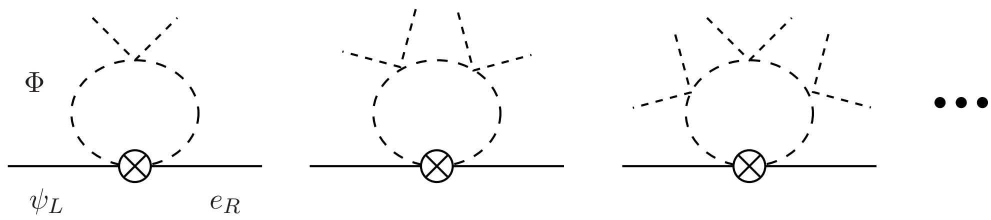

Figure 2. EFT 1-loop diagrams which generate higher-dimensional operators contributing to the charged lepton spectrum. Dashed line represents Φ; $\otimes$ represents the higher-dimensional operator which generate the charged lepton masses at tree-level [corresponding to $\mathcal{O}$ of eq. (2.1)].

Since the above example of underlying mechanism that suppresses higher-dimensional operators is simple, and since suppression of loop-induced $1/\Lambda^n$ corrections within EFT provides a non-trivial cross check of theoretical consistency, we believe that our approach based on EFT has a certain justification and would be useful as a basis for considering more fundamental models.

In the rest of this section, we present a brief overview of the basic ideas of the analysis to be given through sections 3–8, in order to facilitate reading. Our analysis starts from investigating a possibility that the radiative correction generated by a family gauge symmetry cancels the QED correction to Koide's formula (section 3). We find that $\mathrm{U}(3) \simeq \mathrm{SU}(3) \times \mathrm{U}(1)$ family gauge symmetry has a unique property in this regard. In fact, if ψ_L and e_R are assigned to mutually conjugate representations of this symmetry group, the U(3) radiative correction has the same form as the QED correction with opposite sign. In particular, if the gauge coupling of U(3) family symmetry $\alpha_F = g_F^2/(4\pi)$ satisfies the relation $\alpha(m_\tau) \approx \frac{1}{4}\alpha_F(g_F v_3)$, both corrections cancel. We speculate that this relation would be realized within a scenario in which U(3) family gauge symmetry is unified with $\mathrm{SU}(2)_L$ gauge symmetry at 10^2–10^3 TeV scale, although we need to fine tune the unification scale within an accuracy of factor 3.

The non-trivial form of the radiative correction by the U(3) gauge interaction is dictated by the U(3) symmetry and its breaking pattern induced by the VEV $\langle\Phi\rangle$. In particular, multiplicative renormalizability of $\langle\Phi\rangle$ ensures that the correction to Koide's formula is independent of the renormalization scale μ of the effective potential of Φ. Namely, the charged lepton pole masses are determined, up to a common multiplicative constant, directly by the form of the effective potential renormalized at an arbitrary high scale μ ($\leq \Lambda$), and we may ignore the QED and U(3) radiative corrections altogether. For our purpose, it is most convenient to take this scale to be $\mu = \Lambda$. In this part of our analysis, we assume that $\langle\Phi\rangle$ can be brought to a diagonal form by symmetry transformation, and also that Koide's relation for the diagonal elements,

$$\frac{v_1(\mu) + v_2(\mu) + v_3(\mu)}{\sqrt{v_1(\mu)^2 + v_2(\mu)^2 + v_3(\mu)^2}} = \sqrt{\frac{3}{2}}\,, \tag{2.4}$$

is satisfied.

In the second step, we search for an effective potential for which the eigenvalues of $\langle\Phi\rangle$ satisfy the relation (2.4) and reproduce the experimental values of the mass ratios

$v_1 : v_2 : v_3 = \sqrt{m_e} : \sqrt{m_\mu} : \sqrt{m_\tau}$ (sections 4 and 5). If we choose the renormalization scale to be $\mu = \Lambda$, radiative corrections to the effective potential essentially vanish within EFT, or in other words, the form of the effective potential at this scale is determined by physics above the scale Λ as boundary (initial) conditions of EFT. Hence, our goal is to find an effective potential which satisfies the boundary conditions without conflicting symmetry requirements of the theory. Although it may seem an easy task, it still involves fairly non-trivial analyses.

We impose $\mathrm{U}(3) \times \mathrm{SU}(2)$ family symmetry as a symmetry of EFT. The motivation of this choice is that it is the symmetry possessed by the simplest higher-dimensional operator analyzed in the first step. It turns out, however, that this symmetry is not large enough to constrain the form of the effective potential sufficiently. We therefore further assume a symmetry enhancement. Namely, we assume that above the cut-off scale Λ there is an $\mathrm{SU}(9) \times \mathrm{U}(1)$ gauge symmetry, and this symmetry is spontaneously broken to $\mathrm{U}(3) \times \mathrm{SU}(2)$ below the cut-off scale. The symmetry $\mathrm{SU}(9) \times \mathrm{U}(1)$ is motivated by a geometrical interpretation of Koide's relation eq. (2.4). Within this scenario, we still need to introduce an additional scalar field X in order to realize a desirable vacuum configuration. Thus, we analyze the vacuum of the general potential of Φ and X. (Details of the analysis are rather technical.) The conclusion is that in a finite region of the parameter space of the potential, Koide's relation is satisfied by the eigenvalues of $\langle \Phi \rangle$. Furthermore, the eigenvalues can be made consistent with the experimental values of the charged lepton masses without fine tuning of parameters. These are realized in the case that certain hierarchical relations among the parameters of the potential are satisfied, and these relations do not conflict the requirement of the assumed symmetry and symmetry enhancement. We speculate on possible physics scenario above the cut-off scale that may lead to (part of) these hierarchical relations.

So far, these desirable features are satisfied by the eigenvalues of $\langle \Phi \rangle$. There remains, however, a problem that $\langle \Phi \rangle$ cannot be brought to a diagonal form by the $\mathrm{U}(3) \times \mathrm{SU}(2)$ symmetry transformation, and this contradicts the assumption made in the first step. To remedy this difficulty, we introduce yet another field Σ_Y such that, with an appropriate potential with Φ, it can generate an appropriate higher-dimensional operator necessary to produce the charged lepton masses. (Sections 6 and 7.) Although the potential and the higher-dimensional operator involving Σ_Y do not conflict the requirement of the assumed symmetry and symmetry enhancement, these would be the most unsatisfactory part of our model. This is because it is difficult to speculate any plausible scenario above the cut-off scale, which would lead to these potential and higher-dimensional operator.

With all these setups, it is possible to compute the radiative corrections which are induced by the diagrams shown in figure 2. Due to the specific form of the effective potential of Φ and X, corrections to Koide's formula turn out to be quite suppressed, as long as the aforementioned hierarchical conditions between parameters of the potential are satisfied. As already mentioned, this serves as a non-trivial consistency check of the model as an EFT.

There are a few unsolved questions and incompleteness of the present model and these are discussed in sections 8 and 10.

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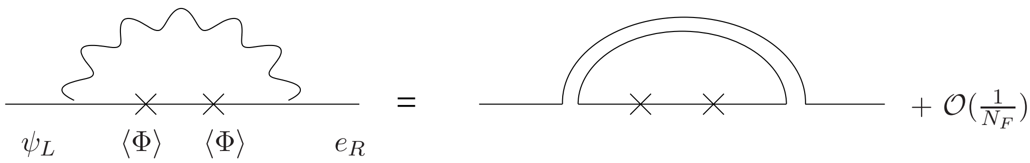

Figure 3. Diagram for the 1-loop correction by the family gauge bosons to the operator $\mathcal{O}$ when both ψ_L and e_R are in the $\mathbf{3}$ of SU(3). The diagram on the right-hand-side shows the flow of family charge in the leading contribution of the $1/N_F$ expansion ($N_F = 3$); closed loop corresponds to $\mathrm{tr}(\langle\Phi\rangle\langle\Phi\rangle)$.

3 Radiative correction by family gauge interaction

In this section we introduce family gauge symmetries and consider radiative corrections to the mass matrix eq. (2.2) by the family gauge interaction. First we consider the case, in which the family gauge group is SU(3) and both ψ_L and e_R are assigned to the $\mathbf{3}$ (fundamental representation) of this symmetry group. We readily see, however, that with this choice of representation, Koide's formula will receive a severe radiative correction unless the family gauge interaction is strongly suppressed. In fact, the 1-loop diagram shown in figure 3 induces an effective operator

$$\mathcal{O}' \sim \frac{\alpha_F}{\pi} \times \kappa\, \bar{\psi}_{Li}\, \varphi\, e_{Ri} \times \frac{\langle\Phi\rangle_{jk}\langle\Phi\rangle_{kj}}{\Lambda^2}\,, \tag{3.1}$$

hence corrections universal to all the charged-lepton masses, $(\delta m_e, \delta m_\mu, \delta m_\tau) \propto (1,1,1)$, are induced. This is due to the fact that the dimension-4 operator $\bar{\psi}_{Li}\, \varphi\, e_{Ri}$ is not prohibited by symmetry. Here, $\alpha_F = g_F^2/(4\pi)$ denotes the gauge coupling constant of the family gauge interaction. As noted above, corrections which are proportional to individual masses do not affect Koide's formula; oppositely, the universal correction violates Koide's formula rather strongly. In order that the correction to Koide's formula cancel the QED correction, a naive estimate shows that α_F/π should be order 10^{-5}, provided that the cut-off Λ is not too large and that the above operator $\mathcal{O}'$ is absent at tree level. If $\mathcal{O}'$ exists at tree level, there should be a fine tuning between the tree-level and 1-loop contributions. The situation is similar if the family symmetry is $O(3)$ and both ψ_L and e_R are in the $\mathbf{3}$, which is also a typical assignment in existing models. In these cases[3] we were unable to find any sensible reasoning for the cancellation between the QED correction and the correction induced by family gauge interaction, other than to regard the cancellation as just a pure coincidence. Hence, we will not investigate these choices of representation further.

In the case that ψ_L is assigned to $\mathbf{3}$ and e_R to $\bar{\mathbf{3}}$ (or *vice versa*) of U(3) family gauge group, (i) the dimension-4 operator $\bar{\psi}_{Li}\, \varphi\, e_{Ri}$ is prohibited by symmetry, and hence corrections universal to all the three masses do not appear; and (ii) marked resemblance of the radiative correction to the QED correction follows. We show these points explicitly in a specific setup.

[3]There are a large number of papers on the fermion flavor structure based on SU(3) or SO(3) family symmetry. See, for instance, [15].

We denote the generators for the fundamental representation of U(3) by T^α ($0 \le \alpha \le 8$), which satisfy

$$\mathrm{tr}\left(T^\alpha T^\beta\right) = \frac{1}{2}\,\delta^{\alpha\beta} \quad ; \quad T^\alpha = T^{\alpha\dagger}\,. \tag{3.2}$$

T^0 is the generator of U(1), hence it is proportional to the identity matrix, while T^a ($1 \le a \le 8$) are the generators of SU(3). Here and hereafter, $\alpha, \beta, \gamma, \dots$ represent U(3) indices $0, \dots, 8$, while $a, b, c, \dots$ represent SU(3) indices $1, \dots, 8$. The explicit forms of T^α are given in appendix A.

We assign ψ_L to the representation $(\mathbf{3}, 1)$, where $\mathbf{3}$ stands for the SU(3) representation and 1 for the U(1) charge, while e_R is assigned to $(\bar{\mathbf{3}}, -1)$. Under U(3), the 9-component field Φ transforms as three $(\mathbf{3}, 1)$'s. Explicitly the transformations of these fields are given by

$$\psi_L \to U\,\psi_L\,, \quad e_R \to U^*\,e_R\,, \quad \Phi \to U\,\Phi \quad ; \quad U = \exp\left(i\theta^\alpha T^\alpha\right)\,, \quad U\,U^\dagger = \mathbf{1}\,. \tag{3.3}$$

We assume that the charged-lepton mass matrix is induced by a higher-dimensional operator $\mathcal{O}^{(\ell)}$ similar to $\mathcal{O}$ in eq. (2.1). We further assume that $\langle\Phi\rangle$ can be brought to a diagonal form in an appropriate basis. Thus, in this basis $\mathcal{O}^{(\ell)}$, after Φ and φ acquire VEVs, turns to the lepton mass terms as

$$\mathcal{O}^{(\ell)} \to \bar{\psi}_L\,\mathcal{M}_\ell^{\rm tree}\,e_R\,, \qquad \mathcal{M}_\ell^{\rm tree} = \begin{pmatrix} m_e^{\rm tree} & 0 & 0 \\ 0 & m_\mu^{\rm tree} & 0 \\ 0 & 0 & m_\tau^{\rm tree} \end{pmatrix} = \frac{\kappa^{(\ell)}(\mu)\,v_{\rm ew}}{\sqrt{2}\Lambda^2}\,\Phi_d(\mu)^2\,, \tag{3.4}$$

where

$$\Phi_d(\mu) = \begin{pmatrix} v_1(\mu) & 0 & 0 \\ 0 & v_2(\mu) & 0 \\ 0 & 0 & v_3(\mu) \end{pmatrix}\,, \qquad v_i(\mu) > 0\,. \tag{3.5}$$

When all v_i are different, U(3) symmetry is completely broken by $\langle\Phi\rangle = \Phi_d$, and the spectrum of the U(3) gauge bosons is determined by Φ_d.

Note that the operator $\mathcal{O}$ in eq. (2.1) is *not* invariant under the U(3) transformations eq. (3.3). As an example of $\mathcal{O}^{(\ell)}$, one may consider

$$\mathcal{O}_1^{(\ell)} = \frac{\kappa^{(\ell)}(\mu)}{\Lambda^2}\,\bar{\psi}_L\,\Phi\,\Phi^T\,\varphi\,e_R\,. \tag{3.6}$$

It is invariant under a larger symmetry U(3) × SU(2), under which Φ transforms as $\Phi \to U\Phi O^T$ ($O\,O^T = \mathbf{1}$). In this case, we need to assume, for instance, that the SU(2) symmetry is gauged and spontaneously broken at a high energy scale before the breakdown of the U(3) symmetry, in order to eliminate massless Nambu-Goldstone bosons and to suppress mixing of the U(3) and SU(2) gauge bosons. A more elaborate example of the higher-dimensional operator, which is consistent with the symmetry and satisfies eqs. (3.4) and (3.5), will be given in section 7. In any case, the properties of $\mathcal{O}^{(\ell)}$ given by eqs. (3.4) and (3.5) are

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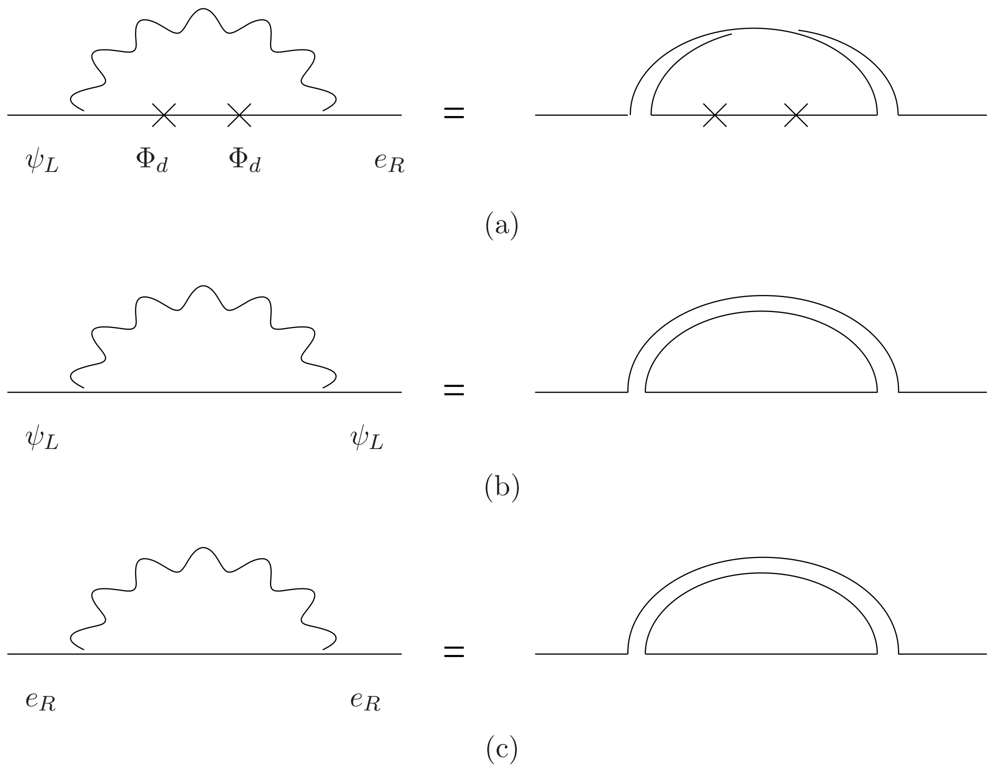

Figure 4. 1-loop diagrams contributing to $\delta m_i^{\rm pole}$ when ψ_L and e_R are in the $(\mathbf{3}, 1)$ and $(\bar{\mathbf{3}}, -1)$, respectively, of SU(3) × U(1). The right-hand-sides show flows of family charge. (a) Correction of the form $\bar{\psi}_L\, \delta\mathcal{M}\, e_R$: charge flow is connected in one line, showing multiplicative renormalization, (b) correction of the form $\bar{\psi}_L\, \not{p}\, Z_\psi\, \psi_L$, and (c) correction of the form $\bar{e}_R\, \not{p}\, Z_e\, e_R$.

sufficient for computing the radiative correction by the U(3) gauge bosons to the mass matrix, without an explicit form of $\mathcal{O}^{(\ell)}$.

We take the U(1) and SU(3) gauge coupling constants to be the same:

$$\alpha_{\rm U(1)} = \alpha_{\rm SU(3)} = \alpha_F\,. \tag{3.7}$$

We compute the radiative correction in Landau gauge, which is known to be convenient for computations in theories with spontaneous symmetry breaking. From the diagrams shown in figures 4(a)(b)(c), we find

$$\delta m_i^{\rm pole} = -\frac{3\,\alpha_F}{8\,\pi}\left[\log\left(\frac{\mu^2}{v_i(\mu)^2}\right) + c\right]\, m_i(\mu)\,, \tag{3.8}$$

$$m_i(\mu) = \frac{\kappa^{(\ell)}(\mu)\, v_{\rm ew}}{\sqrt{2}\Lambda^2}\, v_i(\mu)^2\,. \tag{3.9}$$

Here, c is a constant independent of i. The Wilson coefficient $\kappa^{(\ell)}(\mu)$ is defined in $\overline{\rm MS}$ scheme. $v_i(\mu)$'s are defined as follows: The VEV of Φ at renormalization scale μ, $\Phi_d(\mu) = \langle\Phi(\mu)\rangle$ given by eq. (3.5), is determined by minimizing the 1-loop effective potential in

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Landau gauge (although we do not discuss the explicit form of the effective potential in this section); Φ is renormalized in $\overline{\rm MS}$ scheme. We ignored terms suppressed by $m_i^2/v_j^2(\ll 1)$ in the above expression. Note that the pole mass is renormalization-group invariant and gauge independent. Therefore, the above expression is rendered gauge-independent if we express $v_i(\mu)$ in terms of gauge-independent parameters, such as coupling constants defined in on-shell scheme.

The coefficient of $\log\mu^2$ is determined by the sum of the anomalous dimension of the Wilson coefficient $\kappa^{(\ell)}(\mu)$ and twice of the wave-function renormalization of Φ. (The former is gauge independent, while the latter is not.) The term $\log v_i^2$ originates from the role of the gauge boson masses as an IR cut-off of the loop integral, hence it reflects the spectrum of the gauge bosons. The sign in front of $\log\mu^2$ is opposite to that of the QED correction eq. (1.2), which results from the fact that ψ_L and e_R have the same QED charges but mutually conjugate (opposite) U(3) charges.

In Landau gauge, the diagrams in figures 4(b)(c) are finite and flavor independent, i.e. proportional to δ_{ij} in terms of the family indices; hence they contribute only to the constant c. Apart from this constant, the difference between the QED correction and the correction (3.8) resides in the factors

$$\alpha\,\bar{m}_i\,\log\left(\frac{\mu^2}{\bar{m}_i^2}\right)\delta_{ij}\qquad \text{vs.}\qquad -\,\alpha_F\left[T^\alpha\mathcal{M}_\ell^{\rm tree}\,T^{\beta *}\left\{\log\left(\frac{\mu^2}{M_F^2}\right)\right\}_{\alpha\beta}\right]_{ij} \tag{3.10}$$

in the QED self-energy diagram and the diagram in figure 4(a), respectively. (No sum over i is taken in the former factor.) One may easily identify the factor 2 difference in the coefficients of $\log\mu^2$ using the Fierz identity

$$(T^\alpha)_{ij}\,(T^{\alpha *})_{kl}=(T)^\alpha_{ij}\,(T^\alpha)_{lk}=\frac{1}{2}\,\delta_{ik}\,\delta_{lj}\,. \tag{3.11}$$

From this identity, it follows that the operator $\mathcal{O}^{(\ell)}$ is multiplicatively renormalized; see family charge flow in figure 4(a). $(M_F^2)_{\alpha\beta}$ denotes the mass matrix of the family gauge bosons. After diagonalization, one obtains the spectrum of family gauge bosons as

$$\begin{aligned}\frac{1}{2}\,(M_F^2)_{\alpha\beta}\,f_\mu^\alpha\,f_\mu^\beta &\equiv g_F^2\,{\rm tr}(\Phi_0^\dagger T^\alpha T^\beta\Phi_0)\,f_\mu^\alpha\,f_\mu^\beta \\ &=\frac{g_F^2}{2}\Big[v_1^2\,(\mathcal{F}_\mu^1)^2+v_2^2\,(\mathcal{F}_\mu^1)^2+\frac{1}{2}(v_1^2+v_2^2)\{(\mathcal{F}_\mu^3)^2+(\mathcal{F}_\mu^4)^2\}+v_3^2\,(\mathcal{F}_\mu^5)^2 \\ &\qquad +\frac{1}{2}(v_1^2+v_3^2)\{(\mathcal{F}_\mu^6)^2+(\mathcal{F}_\mu^7)^2\}+\frac{1}{2}(v_2^2+v_3^2)\{(\mathcal{F}_\mu^8)^2+(\mathcal{F}_\mu^9)^2\}\Big]\,.\end{aligned} \tag{3.12}$$

The mass eigenstates $\mathcal{F}_\mu^i$ are labelled in the order of their masses, which are given by

$$\mathcal{F}_\mu^1=\frac{f_\mu^0}{\sqrt{3}}+\frac{f_\mu^3}{\sqrt{2}}+\frac{f_\mu^8}{\sqrt{6}},\qquad \mathcal{F}_\mu^2=\frac{f_\mu^0}{\sqrt{3}}-\frac{f_\mu^3}{\sqrt{2}}+\frac{f_\mu^8}{\sqrt{6}},\qquad \mathcal{F}_\mu^5=\frac{f_\mu^0-\sqrt{2}\,f_\mu^8}{\sqrt{3}}, \tag{3.13}$$

$$\mathcal{F}_\mu^{3,4}=f_\mu^{1,2},\qquad \mathcal{F}_\mu^{6,7}=f_\mu^{4,5},\qquad \mathcal{F}_\mu^{8,9}=f_\mu^{6,7}\,. \tag{3.14}$$

Hence,

$$f_\mu^\alpha T^\alpha = \begin{pmatrix} \frac{1}{\sqrt{2}}\mathcal{F}_\mu^1 & -\frac{i}{2}(\mathcal{F}_\mu^3 + i\mathcal{F}_\mu^4) & -\frac{i}{2}(\mathcal{F}_\mu^6 + i\mathcal{F}_\mu^7) \\ \frac{i}{2}(\mathcal{F}_\mu^3 - i\mathcal{F}_\mu^4) & \frac{1}{\sqrt{2}}\mathcal{F}_\mu^2 & -\frac{i}{2}(\mathcal{F}_\mu^8 + i\mathcal{F}_\mu^9) \\ \frac{i}{2}(\mathcal{F}_\mu^6 - i\mathcal{F}_\mu^7) & \frac{i}{2}(\mathcal{F}_\mu^8 - i\mathcal{F}_\mu^9) & \frac{1}{\sqrt{2}}\mathcal{F}_\mu^5 \end{pmatrix}. \tag{3.15}$$

The form of the radiative correction given by eqs. (3.8) and (3.9) is constrained by symmetries and their breaking patterns. As the diagonal elements of the VEV, $v_3 > v_2 > v_1 > 0$, are successively turned on, gauge symmetry is broken according to the pattern:

$$\mathrm{U}(3) \to \mathrm{U}(2) \to \mathrm{U}(1) \to \text{nothing}\,. \tag{3.16}$$

At each stage, the gauge bosons corresponding to the broken generators acquire masses and decouple. Furthermore, the vacuum Φ_d and the family gauge interaction respect a global $\mathrm{U}(1)_{V1} \times \mathrm{U}(1)_{V2} \times \mathrm{U}(1)_{V3}$ symmetry generated by

$$\psi_L \to U_d\, \psi_L, \;\; e_R \to U_d^*\, e_R, \;\; \Phi_d \to U_d\, \Phi_d\, U_d^* \tag{3.17}$$

with

$$U_d = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} \qquad ; \qquad \phi_i \in \mathbf{R}\,. \tag{3.18}$$

The operator $\mathcal{O}^{(\ell)}$ after symmetry breakdown, eq. (3.4), is not invariant under this transformation but the variation can be absorbed into a redefinition of v_i's. As a result, the lepton mass matrix has a following transformation property:

$$\mathcal{M}_\ell \bigg|_{v_i \to v_i \exp(i\phi_i)} = U_d\, \mathcal{M}_\ell\, U_d^*\,. \tag{3.19}$$

This is satisfied including the 1-loop radiative correction. The symmetry breaking pattern eq. (3.16) and the above transformation property constrain the form of the radiative correction to $\delta m_i^{\rm pole} \propto v_i^2[\log(|v_i|^2) + \text{const.}]$, where the constant is independent of i. Note that $|v_i|^2$ in the argument of logarithm originate from the gauge boson masses, which are invariant under $v_i \to v_i \exp(i\phi_i)$.

The universality of the SU(3) and U(1) gauge couplings eq. (3.7) is necessary to guarantee the above symmetry breaking pattern eq. (3.16). One may worry about validity of the assumption for the universality, since the two couplings are renormalized differently in general. The universality can be ensured approximately if these two symmetry groups are embedded into a simple group down to a scale close to the relevant scale. There are more than one ways to achieve this. A simplest way would be to embed $\mathrm{SU}(3) \times \mathrm{U}(1)$ into SU(4). It is easy to verify that the **4** of SU(4) decomposes into $(\mathbf{3}, -\frac{1}{2}) \oplus (\mathbf{1}, \frac{3}{2})$ under $\mathrm{SU}(3) \times \mathrm{U}(1)$. Hence, the **6** (second-rank antisymmetric representation) and $\bar{\mathbf{6}}$ of SU(4), respectively, include $(\bar{\mathbf{3}}, -1)$ and $(\mathbf{3}, 1)$.

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Within the effective theory under consideration, the QED correction to the pole mass is given just as in eq. (1.2) with $\bar{m}_i(\mu)$ replaced by $m_i(\mu)$. Recall that corrections of the form const. $\times\, m_i$ do not affect Koide's formula. Then, noting $\log v_i^2 = \frac{1}{2}\log m_i^2 + \text{const.}$, one observes that if a relation between the QED and family gauge coupling constants

$$\alpha = \frac{1}{4}\,\alpha_F \tag{3.20}$$

is satisfied, the 1-loop radiative correction induced by family gauge interaction cancels the 1-loop QED correction to Koide's mass formula.

In fact, with the relation (3.20), cancellation holds for all the leading logarithms generated by renormalization group: the coefficient of $\log\mu^2$ of the QED correction is determined by the 1-loop anomalous dimension of the running mass, while the coefficient of $\log\mu^2$ of eq. (3.8) is determined by the anomalous dimension of the Wilson coefficient $\kappa^{(\ell)}$ and twice of the wave-function renormalization of Φ; they are resummed in the same way by 1-loop renormalization group equations. The renormalization group evolution and the symmetry breaking pattern eq. (3.16) in the scale range across the family gauge boson masses dictate how $\log m_i^2$'s induced by family gauge interaction are resummed. The renormalization group evolution and the same symmetry breaking pattern in the QED sector dictate the $\log m_i^2$ resummation of the QED correction, in the scale range across the lepton masses. If $m_i \log m_i^2$ cancel at 1-loop, $\log m_i^2$ dependences in all the leading logarithms $m_i[\alpha\log(\mu^2/m_i^2)]^n$ also cancel. On the other hand, effects of the running of α and α_F do not cancel. It is related to the question which we stated in the Introduction: What are the relevant scales for the coupling constants in the relation (3.20)? The scale of α is determined by the lepton masses, while the scale of α_F is determined by the family gauge boson masses, which should be much higher than the electroweak scale.

Suppose the relation (3.20) is satisfied. Then

$$m_i^{\text{pole}} \propto v_i(\mu)^2 \tag{3.21}$$

holds including the leading logarithms generated by the running of $\kappa^{(\ell)}$ and v_i's. This is valid for any value of μ. This means, if $v_i(\mu)$'s satisfy

$$\frac{v_1(\mu)+v_2(\mu)+v_3(\mu)}{v_0(\mu)} = \sqrt{\frac{3}{2}} \quad ; \quad v_0(\mu) = \sqrt{v_1(\mu)^2+v_2(\mu)^2+v_3(\mu)^2} \tag{3.22}$$

at some scale μ, Koide's formula is satisfied at any scale μ. This is a consequence of the fact that Φ is multiplicatively renormalized. Generally, the form of the effective potential varies with scale μ. If the relation (3.22) is realized at some scale as a consequence of a specific nature of the effective potential (in Landau gauge), the same relation holds automatically at any scale. Although these statements are formally true, physically one should consider scales only above the family gauge boson masses, since decoupling of the gauge bosons is not encoded in $\overline{\text{MS}}$ scheme. For our purpose, it is most appropriate to use eq. (3.21) to relate the charged lepton pole masses with the VEV at the cut-off scale, i.e. $\mu = \Lambda$, which sets a boundary (initial) condition of the effective theory.

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The advantages of choosing Landau gauge in our computation are two folds: (1) The computation of the 1-loop effective potential for the determination of $\langle\Phi\rangle$ becomes particularly simple (as well known in computations of the effective potential in various models); in particular there is no $\mathcal{O}(\alpha_F)$ correction to the effective potential. (2) The lepton wave-function renormalization is finite; as a consequence, the diagrams in figures 4(b)(c) are independent of $\langle\Phi(\mu)\rangle$ and independent of flavor. Due to the former property, there is no $\mathcal{O}(\alpha_F)$ correction to the relation eq. (3.22) if it is satisfied at tree level. Due to the latter property, $\delta m_i^{\rm pole}$ is determined essentially by the diagram in figure 4(a) and a simple relation to $\langle\Phi(\mu)\rangle$ follows.

Let us comment on gauge dependence of our prediction. If we take another gauge and express the radiative correction $\delta m_i^{\rm pole}$ in terms of $\langle\Phi(\mu)\rangle$, the coefficient of $\log(\mu^2/\langle\Phi\rangle^2)$ changes, and other non-trivial flavor dependent corrections are induced. Suppose the relation eq. (3.22) is satisfied at tree level.[4] The VEV $\langle\Phi\rangle$ in another gauge receives an $\mathcal{O}(\alpha_F)$ correction, which induces a correction to eq. (3.22) at $\mathcal{O}(\alpha_F)$. These additional corrections to $\delta m_i^{\rm pole}$ at $\mathcal{O}(\alpha_F)$ should cancel altogether if they are reexpressed in terms of the tree-level v_i's which satisfy eq. (3.22), since the $\mathcal{O}(\alpha_F)$ correction to the relation (3.22) vanishes in Landau gauge. General analyses of gauge dependence of the effective potential may be found in [16, 17].

Now we speculate on a possible scenario how the relation (3.20) may be satisfied. Since the relevant scales involved in α and α_F are very different, we are unable to avoid assuming some accidental factor (or parameter tuning) to achieve this condition. Instead we seek for an indirect evidence which indicates such an accident has occurred in Nature. The relation (3.20) shows that the value of α_F is close to that of the weak gauge coupling constant α_W, since $\sin^2\theta_W(M_W)$ is close to 1/4. In fact, within the SM, $\frac{1}{4}\alpha_W(\mu)$ approximates $\alpha(m_\tau)$ at scale $\mu\sim 10^2$–10^3 TeV. Hence, if the electroweak $\mathrm{SU}(2)_L$ gauge group and the U(3) family gauge group are unified around this scale, naively we expect that

$$\alpha\approx\frac{1}{4}\,\alpha_F \tag{3.23}$$

is satisfied. Since α_W runs relatively slowly in the SM, even if the unification scale is varied within a factor of 3, Koide's mass formula is satisfied within the present experimental accuracy. This shows the level of parameter tuning required in this scenario.

We may generalize our setup and see how the radiative correction alters. If ψ_L and e_R are assigned to $(\mathbf{3},Q_{\psi_L})$ and $(\bar{\mathbf{3}},Q_{e_R})$, respectively, the correction eq. (3.8) generalizes to

$$\delta m_i^{\rm pole}=-\frac{\alpha_F}{8\pi}\left[(Q_{\psi_L}-Q_{e_R}+1)\log\left(\frac{\mu^2}{v_i(\mu)^2}\right)+c'\right]m_i(\mu)\,, \tag{3.24}$$

where c' is a flavor-independent constant. Thus, the form $m_i\log m_i^2$ is maintained. This is not the case if we vary the U(1) charge of Φ, which violates the breaking pattern of

[4] To simplify the argument we consider only those gauges in which tree-level vacuum configuration is gauge independent, such as the class of gauges considered in [16].

gauge symmetry eq. (3.16) strongly.[5] The form $m_i \log m_i^2$ is maintained in yet another generalization, in which $\mathrm{U}(3) \times \mathrm{U}(3)$ symmetry is gauged. We introduce another field $\Sigma : (\mathbf{1}, 0, \mathbf{6}, 2)$ under $\mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{U}(1)$. The symmetry transformations are given by $\psi_L \to U\psi_L$, $e_R \to U^* e_R$, $\Phi \to U\Phi V^\dagger$, $\Sigma \to V\Sigma V^T$ with $U = \exp(i\theta^\alpha T^\alpha)$, $V = \exp(i\tilde{\theta}^\alpha T^\alpha)$. We assume that $\langle\Sigma\rangle = v_\Sigma \mathbf{1}$ with $v_\Sigma \ll v_1, v_2, v_3$, and that the lepton masses are generated by a higher-dimensional operator

$$\mathcal{O}_2^{(\ell)} = \frac{\kappa^{(\ell)}(\mu)}{\Lambda^3}\, \bar{\psi}_L\, \Phi\, \Sigma\, \Phi^T\, \varphi\, e_R\,. \tag{3.25}$$

For the assignment $\psi_L : (\mathbf{3}, 1, \mathbf{1}, 0)$ and $e_R : (\bar{\mathbf{3}}, -1, \mathbf{1}, 0)$, the radiative correction reads

$$\delta m_i^{\rm pole} = -\frac{3}{8\pi} \frac{\alpha_F^4}{\alpha_F^2 + {\alpha'_F}^2} \left[\log\left(\frac{\mu^2}{v_i(\mu)^2} \right) + c'' \right] m_i(\mu)\,, \tag{3.26}$$

where α_F and α'_F denote, respectively, the gauge couplings of the first U(3) and second U(3) symmetries.[6] Thus, the coefficient of $m_i \log m_i^2$ varies in different setups. Accordingly the condition for the cancellation of the QED correction changes from eq. (3.20). We need to seek for other possible scenarios which lead to such conditions, or maybe to let the cancellation be a sheer coincidence. The level of fine tuning required for the coupling(s) is about 1% to meet the present expermental accuracy of Koide's formula.

In the rest of this paper, we do not consider these generalizations. We adhere to eq. (3.20), assuming the scenario in which $\mathrm{SU}(2)_L$ and U(3) gauge symmetries are unified at around 10^2–10^3 TeV. In this paper we do not construct a model which incorporates this unification scenario. We simply assume that this unification scenario is realized in the underlying full theory, in which the unification scale is at or around the cut-off scale Λ of our effective theory; we further assume that the hierarchy between v_3 and $\Lambda(> v_3)$ is mild; see discussions in sections 2 and 8.

4 Potential minimum and charged lepton spectrum

The analysis in the previous section indicates relevance of the U(3) family gauge symmetry in relation to the charged lepton spectrum and Koide's mass formula. In this section we study the potential of Φ invariant under this family symmetry and its classical vacuum. In particular, we propose a mechanism for generating a realistic charged lepton spectrum, assuming that Koide's mass relation is protected. For later convenience, we express components of Φ using T^α, defined in eqs. (3.2) and (A.1), as the basis:

$$\Phi = \Phi^\alpha\, T^\alpha\,. \tag{4.1}$$

In general Φ^α takes a complex value.

[5] Even in the case in which only the U(1) charges of ψ_L and e_R are varied, this symmetry breaking pattern is violated but only softly through the gauge interaction of ψ_L and e_R. By contrast, varying the U(1) charge of Φ affects the spectrum of the gauge bosons.

[6] If $\alpha_F = \alpha'_F$, $\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)_A$ can be embedded into SU(6). In this case, ψ_L and e_R can be assigned to the $\mathbf{6}$ and $\bar{\mathbf{6}}$ of SU(6), respectively. The remaining $\mathrm{U}(1)_V$, corresponding to the lepton number, is unbroken, so it may be taken as a global symmetry.

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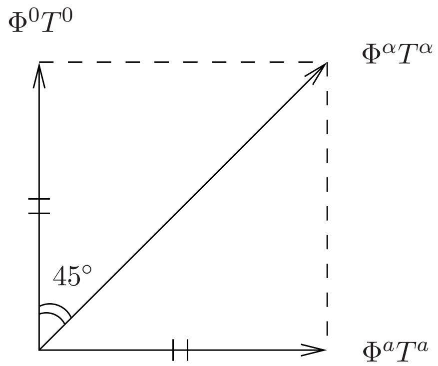

Figure 5. Geometrical interpretation of eq. (4.2). Eq. (3.2) defines the inner product in a 9-dimensional real vector space spanned by the basis $\{T^\alpha\}$. Since $\Phi^0 T^0$, $\Phi^a T^a$ and $\Phi = \Phi^\alpha T^\alpha$ form an isosceles right triangle, the angle between T^0 and Φ is 45°. This is Koide's formula in the basis where Φ is diagonal [3].

The largest symmetry that can be imposed on the higher-dimensional operator $\mathcal{O}^{(\ell)}$ is $\mathrm{U}(3) \times \mathrm{U}(3)$. [An example is given in eq. (3.25).] We may consider the potential of Φ consistent with this symmetry, allowing only operators with dimension 4 or less. A general analysis shows that, for any choice of the parameters (couplings) of this potential, the classical vacuum $\langle \Phi \rangle$, after its diagonalization, does not satisfy the relation (3.22) [18]. Namely, there is no classical vacuum that leads to Koide's mass formula. If we impose a smaller symmetry on the potential of Φ, it is possible to tune the parameters in the potential and realize the relation (3.22) as well as a realistic charged lepton spectrum. We were, however, unable to find a sensible reasoning for tuning the parameters with an accuracy necessary to realize Koide's mass formula.

We may reverse the argument partially and search for a realistic vacuum within a restricted set of configurations. Namely, in view of the high accuracy with which Koide's mass formula is realized in Nature, it may make sense to assume that this mass relation is protected by some mechanism. (An example of such a mechanism will be given in the next section.) We assume that the vacuum configuration satisfies

$$(\Phi^0)^2 = \Phi^a \, \Phi^a \qquad ; \qquad \Phi^\alpha \in \mathbf{R} \tag{4.2}$$

in an appropriate basis allowed by the symmetry. In this case, the relation (3.22) is satisfied by the eigenvalues of Φ [10]; see figure 5. Then we minimize the potential of Φ within the configurations which satisfy this condition. Eq. (4.2) imposes one condition among the three masses of leptons. Apart from the overall normalization of the spectrum, there remains only one free parameter, which should be fixed by minimizing the potential. Since the condition (4.2) or the relation (3.22) treats the three mass eigenvalues symmetrically, *a priori* it seems difficult to generate a hierarchical spectrum. If we impose the $\mathrm{U}(3) \times \mathrm{U}(3)$ symmetry to the potential of Φ, there is no vacuum corresponding to a realistic lepton spectrum. We find that in the case of the $\mathrm{U}(3) \times \mathrm{SU}(2)$ symmetry, a realistic spectrum follows from the vacuum of a simple potential.

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In the rest of this section we study a classical vacuum of the potential of Φ which is invariant under the $\mathrm{U}(3)\times\mathrm{SU}(2)$ transformation

$$\Phi \to U\,\Phi\,O^T \quad ; \quad U\,U^\dagger = O\,O^T = \mathbf{1}\,. \tag{4.3}$$

Up to dimension 4, there are only 4 independent invariant operators. We parametrize the potential as

$$V(\Phi) = V_{\Phi 1}(\Phi) + V_{\Phi 2}(\Phi) + V_{\Phi 3}(\Phi) \tag{4.4}$$

where

$$V_{\Phi 1}(\Phi) = \lambda\left[\mathrm{tr}(\Phi^\dagger\,\Phi) - v^2\right]^2 , \tag{4.5}$$

$$V_{\Phi 2}(\Phi) = \varepsilon_{\Phi 2}\,\mathrm{tr}(\Phi^\dagger\,\Phi\,\Phi^\dagger\,\Phi)\,, \tag{4.6}$$

$$V_{\Phi 3}(\Phi) = \varepsilon_{\Phi 3}\,\mathrm{tr}(\Phi\,\Phi^T\,\Phi^*\,\Phi^\dagger)\,. \tag{4.7}$$

The 4 independent parameters λ, v^2, $\varepsilon_{\Phi 2}$ and $\varepsilon_{\Phi 3}$ are real. These potentials are classified according to the symmetries: Since $\mathrm{tr}(\Phi^\dagger\,\Phi) = \frac{1}{2}\,\Phi^{\alpha *}\Phi^\alpha$, $V_{\Phi 1}$ is invariant under SO(18); $V_{\Phi 2}$ is invariant under $\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)$; $V_{\Phi 3}$ is invariant under $\mathrm{U}(3)\times\mathrm{SU}(2)\simeq\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$.

We assume the condition (4.2). When $\varepsilon_{\Phi 2}=0$ and $\lambda, v^2, \varepsilon_{\Phi 3} > 0$, the configuration which minimizes $V(\Phi)$ under this condition corresponds to a charged lepton spectrum very close to the experimentally observed one. Let us describe the details of this configuration. Using the transformation (4.3), any Φ can be brought to a form parametrized by 6 real parameters. Without loss of generality, we can choose $(\Phi^0, \Phi^2, \Phi^3, \Phi^4, \Phi^6, \Phi^8)$ as the real parameters, while Φ^1, Φ^5, Φ^7 are set to zero.[7] Then it is straightforward (but cumbersome) to minimize the potential $V_{\Phi 1}+V_{\Phi 3}$ under the condition (4.2). One finds the global minimum at the configuration

$$\Phi_0 : \begin{cases} (\Phi_0^0, \Phi_0^2, \Phi_0^8) = v_0\,(1, \sqrt{1-x_0^2}, x_0) \\ \text{other } \Phi_0^\alpha = 0 \end{cases} \tag{4.8}$$

where[8]

$$x_0 = \frac{(\sqrt{129}+9)^{1/3} - (\sqrt{129}-9)^{1/3}}{2^{7/6}\cdot 3^{2/3}} = 0.2997\ldots\,, \tag{4.9}$$

$$v_0 = v\left[1 + \frac{1-3\sqrt{2}x_0 + 4x_0^2}{24}\,\frac{\varepsilon_{\Phi 3}}{\lambda}\right]^{-\frac{1}{2}} \approx \frac{v}{\sqrt{1+0.003656\,(\varepsilon_{\Phi 3}/\lambda)}}\,. \tag{4.10}$$

[7] Since any Φ can be diagonalized by a bi-unitary transformation, $\Phi_d = U\Phi V^\dagger$, Φ can be brought to an hermite matrix by a U(3) transformation as $\Phi' = V^\dagger U\Phi = V^\dagger\Phi_d V$, i.e. $\Phi^{\alpha\prime}\in\mathbf{R}$. Noting that $(\Phi^{2\prime}, \Phi^{5\prime}, \Phi^{7\prime})$ transforms as the **3** of the diagonal subgroup $\mathrm{SU}(2)_V\subset\mathrm{U}(3)\times\mathrm{SU}(2)$, we may set $\Phi^{5\prime}, \Phi^{7\prime}=0$ using this transformation. Using a residual degree of freedom, which rotates $(\Phi^{1\prime}, \Phi^{3\prime})$ as a real doublet of $O(2)$, we can set $\Phi^{1\prime}=0$.

[8] x_0 is a real solution to the equation $8\,x^3 + 4\,x - \sqrt{2} = 0$.

There are no other degenerate vacua except those which are connected to Φ_0 by the $\mathrm{U}(3)\times\mathrm{SU}(2)$ transformation (4.3). Note that there is a residual $\mathrm{U}(1)_{T^2}$ symmetry corresponding to the transformation

$$\Phi \to \exp(i\theta T^2)\,\Phi\,\exp(-i\theta T^2)\,, \tag{4.11}$$

which keeps the above vacuum invariant. This is a subgroup of $\mathrm{U}(3)\times\mathrm{SU}(2)$.

The three eigenvalues of Φ_0 are given by

$$\begin{aligned}(v_1,v_2,v_3) &= \frac{v_0}{6}\left(\sqrt{6}+\sqrt{3}\,x_0-3\sqrt{1-x_0^2},\ \sqrt{6}-2\sqrt{3}\,x_0,\ \sqrt{6}+\sqrt{3}\,x_0+3\sqrt{1-x_0^2}\right)\\ &\approx v_0\,(0.01775, 0.2352, 0.9718)\,.\end{aligned} \tag{4.12}$$

The corresponding experimental values read

$$(\sqrt{m_e},\sqrt{m_\mu},\sqrt{m_\tau}) \approx \sqrt{m_\Sigma}\,(0.01647, 0.2369, 0.9714)\,, \tag{4.13}$$

where $m_\Sigma = m_e + m_\mu + m_\tau$. We pay particular attention to the value of v_3/v_0, which approximates the corresponding experimental value with an accuracy of 4×10^{-4}. Since the constraint (4.2) treats the three eigenvalues symmetrically, some kind of fine tuning should be inherent in this vacuum configuration corresponding to a hierarchical spectrum. Indeed, this is reflected to the fact that, if the value of v_3/v_0 is varied slightly from the above value under the condition (4.2), variations of v_1/v_0 and v_2/v_0 are fairly enhanced. (Note that the values of v_1/v_0 and v_2/v_0 are fixed by v_3/v_0.) As a result, a tiny perturbation to the potential can bring all v_i/v_0 to be consistent with the experimental values. For instance, turning on $V_{\Phi 2}$ with $\varepsilon_{\Phi 2}/\varepsilon_{\Phi 3}\approx -6\times10^{-3}$ will achieve this. This feature is indifferent to details of perturbations: they can be any mixture of $V_{\Phi 2}$, higher-dimensional operators, and radiatively induced potentials (log potentials).

The following comparison may illustrate markedness of the above configuration. When this configuration is the zeroth-order vacuum, correct orders of magnitude of m_e/m_Σ, m_μ/m_Σ, m_τ/m_Σ are reproduced if perturbations are sufficiently small. By contrast, when the zeroth-order value of v_3/v_0 is in less accurate agreement with the experimental value, a fine tuning of perturbative contributions is necessary even to reproduce the mass ratios m_i/m_Σ with correct orders of magnitude.[9]

We find it quite intriguing that the vacuum of such a simple potential, which respects the U(3) family symmetry, selects this particular value of v_3/v_0 very close to the realistic value. Noting the relation (3.21) between the lepton pole masses and the VEV of Φ at high energy scales, the above feature may suggest that the potential takes a form $V(\Phi)\approx V_{\Phi 1}(\Phi)+V_{\Phi 3}(\Phi)$ at the cut-off scale $\mu=\Lambda$.

At this stage, it is unclear what mechanism protects the condition (4.2). Furthermore, it is unclear why $V_{\Phi 2}$ should be so much suppressed compared to $V_{\Phi 3}$, $|\varepsilon_{\Phi 2}/\varepsilon_{\Phi 3}|\lesssim 10^{-2}$. Naively, one would expect that radiative corrections induce $V_{\Phi 2}$ at least with a similar order of magnitude as $V_{\Phi 3}$. In the next section, we will present a possible mechanism or scenario to solve these problems (not completely but at least in such a way to circumvent fine tuning of parameters).

[9] For instance, if $v_3/v_0\approx 0.9856$ (1.5% difference from the experimental value), m_e and m_μ are predicted to be the same, $m_e/m_\Sigma = m_\mu/m_\Sigma$.

5 A minimal potential

In this section, we present a potential of scalar fields, possibly minimal in its content, which realizes $\langle\Phi\rangle \approx \Phi_0$, defined in eq. (4.8), at its classical vacuum. This is discussed within an effective theory which has U(3) $\times$ SU(2) family gauge symmetry, valid below the cut-off scale Λ. The assignment to SU(3)$\times$SU(2)$\times$U(1) of the fields, which are already introduced in the previous sections, reads

$$\psi_L : (\mathbf{3}, \mathbf{1}, 1), \qquad e_R : (\bar{\mathbf{3}}, \mathbf{1}, -1), \qquad \Phi : (\mathbf{3}, \mathbf{3}, 1), \qquad \varphi : (\mathbf{1}, \mathbf{1}, 0), \tag{5.1}$$

with the transformation properties

$$\psi_L \to U\,\psi_L, \qquad e_R \to U^*\,e_R, \qquad \Phi \to U\,\Phi\,O^T, \qquad \varphi \to \varphi\,, \tag{5.2}$$

$$U = \exp(i\theta^\alpha T^\alpha)\,, \quad O = \exp(2i\tilde{\theta}^x T^x) \;\; (x = 2, 5, 7) \quad ; \quad U\,U^\dagger = O\,O^T = \mathbf{1}\,. \tag{5.3}$$

Furthermore, we assume that above the cut-off scale Λ there is an SU(9) $\times$ U(1) gauge symmetry and that this symmetry is spontaneously broken to U(3) $\times$ SU(2) below the cut-off scale.

Let us describe the assignment of the fields to the group SU(9) $\times$ U(1). Φ is assigned to $(\mathbf{9}, 1)$; its transformation is given by $\Phi^\alpha \to \tilde{U}^{\alpha\beta}\Phi^\beta$ with a 9-by-9 unitary matrix[10] $\tilde{U}^{\alpha\beta}$. ψ_L is included in $(\overline{\mathbf{36}}, 1)$ (the $\mathbf{36}$ is the second-rank antisymmetric representation), which decomposes into $(\bar{\mathbf{6}}, \mathbf{3}, 1) \oplus (\mathbf{3}, \mathbf{1}, 1) \oplus (\mathbf{3}, \mathbf{5}, 1)$ after the symmetry breakdown; similarly e_R is included in $(\mathbf{36}, -1)$. φ is a singlet under SU(9) $\times$ U(1).

In order to realize a desirable vacuum configuration, we introduce another field X, which is in the representation $(\mathbf{45}, Q_X)$ (the $\mathbf{45}$ is the second-rank symmetric representation) and is unitary. It can be represented by a 9-by-9 unitary symmetric matrix:

$$X^{\alpha\beta} = X^{\beta\alpha}, \qquad X^{\alpha\gamma}\,X^{\beta\gamma *} = \delta^{\alpha\beta} \quad ; \quad X^{\alpha\beta} \to \tilde{U}^{\alpha\rho}\,X^{\rho\sigma}\,\tilde{U}^{\beta\sigma}\,. \tag{5.4}$$

X decomposes into $X_S^1(\mathbf{6}, \mathbf{1}, Q_X) \oplus X_S^5(\mathbf{6}, \mathbf{5}, Q_X) \oplus X_A(\bar{\mathbf{3}}, \mathbf{3}, Q_X)$ after the symmetry breakdown. See appendix B for the decomposition of X under U(3) $\times$ SU(2).

We may summarize the essence of how to realize a vacuum, which satisfies eq. (4.2), as follows. If the VEV of X takes a value

$$X_0^{\alpha\beta} = [\mathrm{diag.}(-1, +1, \cdots, +1)]_{\alpha\beta} = -2\,\delta^{\alpha 0}\,\delta^{\beta 0} + \delta^{\alpha\beta}\,, \tag{5.5}$$

an SU(9) $\times$ U(1)-invariant condition

$$\Phi^\alpha\,X^{\alpha\beta *}\,\Phi^\beta = 0 \tag{5.6}$$

reduces to the first condition in eq. (4.2) at $X = X_0$. The second condition in eq. (4.2) can be realized by maximizing $|\Phi^0|^2$ upon fixing the value of $\Phi^{\alpha *}\Phi^\alpha$ and imposing the first condition of eq. (4.2); see appendix C.1. These conditions can be met at the classical vacuum of the potential of the scalar fields under consideration, with an appropriate choice

[10] For instance, $2\,\mathrm{tr}(\Phi^\dagger\Phi) = \Phi^{\alpha *}\Phi^\alpha$ is invariant under SU(3) $\times$ SU(2) $\times$ U(1) as well as SU(9) $\times$ U(1).

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of parameters in the potential. We may avoid fine tuning of the parameters, except for the one related to stabilization of the electroweak scale.

In what follows we do not discuss any details of the theory above the scale Λ. Rather we use general properties of $\mathrm{SU}(9) \times \mathrm{U}(1)$ gauge symmetry to infer boundary conditions to be imposed at the scale Λ. We also investigate boundary conditions at this scale required from the low-energy side phenomenologically, consistently with symmetry requirements.

We study the potential and its vacuum of the scalar fields, Φ, X and φ. First we analyze the potential of a specific form (or with a specific choice of parameters of the potential), which incorporates an essential part of our model. Later we extend the potential to more general forms. The potential we analyze reads

$$V(\Phi, X) = V_{\Phi 1} + V_{\Phi 3} + V_{X1} + V_{K1} + V_{\Phi X1}\,, \tag{5.7}$$

where $V_{\Phi 1}$ and $V_{\Phi 3}$ are defined in eqs. (4.5) and (4.7), respectively, and the other potentials are defined by

$$\begin{aligned}
V_{X1} &= \varepsilon_{X1}\, v^4\, \mathrm{tr}(T^\alpha\, T^\rho\, T^\beta\, T^\sigma)\, X^{\alpha\beta}\, X^{\rho\sigma *}\,, && (5.8)\\
V_{K1} &= \varepsilon_{K1}\, \left|\Phi^\alpha\, X^{\alpha\beta *}\, \Phi^\beta\right|^2, && (5.9)\\
V_{\Phi X1} &= -\varepsilon_{\Phi X1}\, v^2\, \mathrm{tr}(T^\alpha\, T^\beta\, \Phi^\dagger\, T^\rho\, T^\sigma\, \Phi)\, X^{\alpha\sigma *}\, X^{\beta\rho}\,. && (5.10)
\end{aligned}$$

All the parameters of the potential, λ, $\varepsilon_{\Phi 3}$, ε_{X1}, ε_K, $\varepsilon_{\Phi X1}$, v, are taken to be positive. Note that since the field X is unitary, it is dimensionless. The physical scale of its VEV is determined by the kinetic term of X, which is normalized as $f_X^2\, |(D_\mu X)^{\alpha\beta}|^2$. Thus, the physical scale of the VEV of X is $\mathcal{O}(f_X)$. We choose f_X to be much smaller than v (the scale of $\langle\Phi\rangle$), such that the spectrum of the family gauge bosons is determined predominantly by $\langle\Phi\rangle$. (See discussion in section 8.)

One may verify the following properties of the potential:

- The global minimum of V_{X1} is at $X = X_0$, defined by eq. (5.5). Degenerate configurations are only those which are connected to X_0 by the $\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)$ transformation (the symmetry transformation of V_{X1}).

- V_{K1} is minimized if eq. (5.6) holds. This equation reduces to the condition $(\Phi^0)^2 = \Phi^a \Phi^a$ in the case that $X = X_0$.

- If $X = X_0$, $V_{\Phi X1} \sim -\,\varepsilon_{\Phi X1} |\Phi^0|^2$ up to a term that can be absorbed in $V_{\Phi 1}$:

$$V_{\Phi X1}\Big|_{X=X_0} = -\varepsilon_{\Phi X1}\, v^2\, \left[\frac{5}{8}\,|\Phi^0|^2 + \frac{1}{18}\,\Phi^{\alpha *}\Phi^\alpha\right]. \tag{5.11}$$

- If the constraints $(\Phi^0)^2 = \Phi^a \Phi^a$ and $\Phi^{\alpha *}\Phi^\alpha = 2v_0^2 (> 0)$ are imposed, $V_{\Phi X1}|_{X=X_0}$ is minimized when Φ^α's have a common phase $\theta \in \mathbf{R}$, namely $e^{-i\theta}\Phi^\alpha \in \mathbf{R}$ for all α; see appendix C.1.

- $\Phi = \Phi_0$, defined by eq. (4.8), is the classical vacuum of $V_{\Phi 3}$ under the constraints $\Phi^{\alpha *}\Phi^\alpha = 2v_0^2$, $(\Phi^0)^2 = \Phi^a\Phi^a$ and $\Phi^\alpha \in \mathbf{R}$.

- If the constraints $(\Phi^0)^2 = \Phi^a\Phi^a$ and $\Phi^{\alpha *}\Phi^\alpha = 2v_0^2$ are imposed, the first derivative of $V_{\Phi 3}$ vanishes,[11] $\partial V_{\Phi 3}/\partial \Phi^\alpha = \partial V_{\Phi 3}/\partial \Phi^{\alpha *} = 0$, at $\Phi = \Phi_0$. This is not trivial: In general there may be a non-zero derivative in an imaginary direction, since Φ_0 is determined assuming $\Phi^\alpha \in \mathbf{R}$.

- All terms in $V(\Phi, X)$ except $V_{\Phi 3}$ is invariant under $\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1) (\supset \mathrm{U}(3) \times \mathrm{SU}(2))$, while the variation of $V_{\Phi 3}$ is positive semi-definite at $\Phi = \Phi_0$. Namely, if $\Phi_0' = U_1 \Phi_0 U_2^\dagger$ $(U_1U_1^\dagger = U_2U_2^\dagger = \mathbf{1})$, $V_{\Phi 3}(\Phi_0') \geq V_{\Phi 3}(\Phi_0)$; see appendix C.2.

Due to these properties, the classical vacuum of $V(\Phi, X)$ in the limit $\varepsilon_{\Phi 3}, \varepsilon_{\Phi X1} \ll \varepsilon_{K1}, \varepsilon_{X1}$ is given by $\Phi = e^{i\theta}\Phi_0$ and $X = X_0$ up to a $\mathrm{U}(3) \times \mathrm{SU}(2)$ transformation, provided that $\varepsilon_{\Phi X1}/\varepsilon_{\Phi 3}$ exceeds a critical value to assure the reality condition on Φ^α (up to a common phase):

$$\frac{\varepsilon_{\Phi X1}}{\varepsilon_{\Phi 3}} > 0.02164\dots . \tag{5.12}$$

We note that the definition of v_0 should be modified, including the effect of $V_{\Phi X1}$, from eq. (4.10) to

$$\begin{aligned} v_0 &= v \left[1 + \frac{53}{144}\,\frac{\varepsilon_{\Phi X1}}{\lambda} \right]^{\frac{1}{2}} \left[1 + \frac{1 - 3\sqrt{2}x_0 + 4x_0^2}{24}\,\frac{\varepsilon_{\Phi 3}}{\lambda} \right]^{-\frac{1}{2}} \\ &\approx v\sqrt{\frac{1 + 0.3681(\varepsilon_{\Phi X1}/\lambda)}{1 + 0.003656\,(\varepsilon_{\Phi 3}/\lambda)}} . \end{aligned} \tag{5.13}$$

The operators $V_{\Phi 3}$ and $V_{\Phi X1}$, whose couplings need to be suppressed, are non-invariant under $\mathrm{SU}(9) \times \mathrm{U}(1)$. This is a key property of our model which allows us to circumvent fine tuning, as we will discuss shortly. As far as the charged lepton masses are concerned, the phase θ can be removed by redefining the phases of ψ_L and e_R. Hence, we set $\theta = 0$ in the following analysis for simplicity.[12]

The $\mathcal{O}(\varepsilon_{\Phi 3})$ and $\mathcal{O}(\varepsilon_{\Phi X1})$ corrections to the vacuum configuration can be computed.

[11] It can be shown, for instance, from the invariance of both $V_{\Phi 3}$ and Φ_0 under the $\mathrm{U}(1)_{T^2}$ transformation eq. (4.11) and two Z_2 transformations given by $\Phi^\alpha \to (P_i^{\alpha\beta}\Phi^\beta)^*$ with $P_1 = \mathrm{diag.}(+1, -1, +1, +1, -1, +1, +1, -1, +1)$ and $P_2 = \mathrm{diag.}(+1, -1, +1, +1, +1, -1, -1, +1, +1)$.

[12] The degeneracy of the vacua parametrized by θ originates from an accidental $\mathrm{U}(1)_\Phi$ global symmetry of the potential $V(\Phi, X)$, under which the overall phase of Φ is rotated independently of X. The degeneracy will be lifted if we include in the potential operators which break this accidental symmetry.

In an appropriate basis, these are given by

$$\delta\Phi^0 = v_0 \left[\left(\frac{1}{16\,\varepsilon_{K1}} + \frac{3}{13\,\varepsilon_{X1}} \right) \bar{\varepsilon}_\Phi + \frac{6-5\sqrt{2}\,x_0}{78\,\varepsilon_{X1}} \varepsilon_{\Phi X1} \right], \tag{5.14}$$

$$\delta\Phi^2 = -\sqrt{1-x_0^2}\,\delta\Phi^0, \qquad \delta\Phi^8 = -x_0\,\delta\Phi^0, \tag{5.15}$$

$$\delta X^{02} = \delta X^{20} = 2\sqrt{1-x_0^2} \left(\frac{3}{13\,\varepsilon_{X1}} \bar{\varepsilon}_\Phi + \frac{5-2\sqrt{2}\,x_0}{78\,\varepsilon_{X1}} \varepsilon_{\Phi X1} \right), \tag{5.16}$$

$$\delta X^{08} = \delta X^{80} = 2\,x_0 \left(\frac{3}{13\,\varepsilon_{X1}} \bar{\varepsilon}_\Phi + \frac{1+2\sqrt{2}\,x_0 - 8\,x_0^2}{78\,\varepsilon_{X1}} \varepsilon_{\Phi X1} \right), \tag{5.17}$$

$$\text{all other } \;\delta\Phi^\alpha,\, \delta X^{\alpha\beta} = 0\,, \tag{5.18}$$

where

$$\bar{\varepsilon}_\Phi = \frac{5}{8}\,\varepsilon_{\Phi X1} - \frac{\sqrt{6}\,v_1 v_3\,(v_1+v_3)}{v_0^3}\,\varepsilon_{\Phi 3} \approx \frac{5}{8}\,(\varepsilon_{\Phi X1} - 0.06690\,\varepsilon_{\Phi 3})\,, \tag{5.19}$$

and v_i's are given by eq. (4.12). Hence, violation of Koide's mass formula is expected to be $\mathcal{O}(\varepsilon_\Phi/\varepsilon_{K1})$ or $\mathcal{O}(\varepsilon_\Phi/\varepsilon_{X1})$, where ε_Φ represents $\varepsilon_{\Phi X1}$ or $\varepsilon_{\Phi 3}$. The explicit expression of the charged lepton spectrum including the above corrections depends on the precise form of the higher-dimensional operator $\mathcal{O}^{(\ell)}$ which generates the lepton masses. Naively one expects that $\bar{\varepsilon}_\Phi/\varepsilon_{K1}$, $\bar{\varepsilon}_\Phi/\varepsilon_{X1}$, $\varepsilon_{\Phi X}/\varepsilon_{X1} \lesssim 10^{-5}$ should be satisfied, in order to meet the experimental accuracy of Koide's formula. [Compare with the estimates below eq. (7.10).]

Next we consider the potential of Φ and X in general and examine conditions necessary for realizing $\langle\Phi\rangle \approx \Phi_0$ and $\langle X\rangle \approx X_0$. Noting that X is unitary, the potential invariant under $SU(9) \times U(1)$ can be written as

$$V_{\Phi X}^{SU(9)\times U(1)} = \sum_{n,m\geq 0} C_{nm}\,(\Phi^{\alpha *}\Phi^\alpha)^n \left|\Phi^\beta\, X^{\beta\gamma *}\,\Phi^\gamma\right|^{2m}. \tag{5.20}$$

When the coefficients C_{nm} are appropriately chosen (without fine tuning), $V_{\Phi X}^{SU(9)\times U(1)}$ can have a minimum at $\Phi^{\alpha *}\Phi^\alpha > 0$ and $\Phi^\beta\, X^{\beta\gamma *}\,\Phi^\gamma = 0$. These are satisfied by $\Phi = \Phi_0$ and $X = X_0$.

All the other operators are non-invariant under $SU(9) \times U(1)$. We separate them into three categories: those which depend only on X (V_X), those which depend only on Φ (V_Φ), and those which depend on both Φ and X ($V_{\Phi X}$). Requirements to each of them are as follows:

- We can show that the first derivative of V_X vanishes at $X = X_0$ if CP invariance is preserved; see appendix C.4. This means that, assuming CP invariance, $X = X_0$ can be the global minimum of V_X in a certain domain of the parameter space (spanned by the parameters in V_X), without fine tuning of parameters.[13]

[13] One example is the case in which V_{X1} gives a dominant contribution in V_X, although other operators need not be suppressed by orders of magnitude. This is because, contributions of other operators cannot create a non-zero derivative at $X = X_0$, and the position of the global minimum is altered only if their contributions are large enough to create a global minimum at another configuration.

- Up to dimension 4, V_Φ consists only of $V_{\Phi 2}$ and $V_{\Phi 3}$; see section 4. Since effects of higher-dimensional operators are expected to be suppressed, V_Φ will be minimized at $\Phi \approx \Phi_0$ if $\varepsilon_{\Phi 2} \ll \varepsilon_{\Phi 3} \ll \varepsilon_K, \varepsilon_X$ and $\Phi^\alpha\, X_0^{\alpha\beta}\, \Phi^\beta = 0$. Here, ε_K/v and ε_X/v represent typical magnitudes of the second derivatives of $V_{\Phi X}^{\mathrm{SU}(9)\times \mathrm{U}(1)}$ and V_X, respectively, at their minima.

- The contribution of $V_{\Phi X}$ needs to be suppressed as compared to those of $V_{\Phi X}^{\mathrm{SU}(9)\times \mathrm{U}(1)}$ and V_X. If we can treat $V_{\Phi X}$ as a perturbation, we may substitute $X = X_0$ and $\Phi^{\alpha *}\Phi^\alpha = v_0^2$ in the lowest-order approximation. Then $V_{\Phi X}$ becomes dependent only on Φ^0 and Φ^x ($x = 2, 5, 7$). The role of the operators dependent only on Φ^0 is similar to $V_{\Phi X1}$; their total contribution should not be too small compared to that of $V_{\Phi 3}$ and should enforce the reality condition on Φ^α; c.f. eq. (5.12). The role of the operators dependent on Φ^x is similar to $V_{\Phi 2}$; in order to suppress corrections to the lepton spectrum, contributions of these operators need to be suppressed compared to that of $V_{\Phi 3}$.

Thus, under appropriate conditions, $\langle \Phi \rangle \approx \Phi_0$ can be realized with a more general potential than the specific potential eq. (5.7). Coefficients of certain operators need to be suppressed compared to the others. One may estimate typical orders of magnitudes of hierarchies required in the constraints, from the analysis of the specific potential $V(\Phi, X)$, which serves as a reference case. Moreover, in principle it is straightforward to compute corrections to the vacuum configuration similar to eqs. (5.14)–(5.18) for a more general potential.

Let us comment on CP invariance. We may assume that either CP invariance is broken explicitly (but weakly) or it is broken spontaneously. In the former case, since there is no observed CP asymmetry in the lepton sector, we may assume effects of the explict breaking are very small and will not affect our argument given above significantly. In the latter case, since CP asymmetry resides only in the Yukawa interaction in the SM, we may attribute the Kobayashi-Maskawa CP phase to the VEV of the scalar field, which is presumably existent to give masses to the quarks, while keeping all the interactions in the $\mathrm{U}(3)\times \mathrm{SU}(2)$ effective theory CP-invariant. As we do not discuss quark sector at all in this paper, this argument is rather ambiguous. In passing, we note that all the operators in the potential $V(\Phi, X)$ [eq. (5.7)] are CP-invariant; see appendix C.3 for the CP transformations.

Furthermore, the Higgs field φ needs to be incorporated in the potential. Since φ is a singlet under $\mathrm{U}(3) \times \mathrm{SU}(2)$, it can be included effectively by replacing the coefficients of the operators in the above discussion by functions (polynomials) of $\varphi^\dagger\varphi$, e.g. $C_{nm} \to C_{nm}(\varphi^\dagger\varphi)$. Hence, the conditions on the coefficients are the same as above when evaluated at $\varphi^\dagger\varphi = v_{\mathrm{ew}}^2 (\ll v_0^2)$. On the other hand, the VEV of φ is determined from the same potential after substituting $\Phi \approx \Phi_0$ and $X \approx X_0$, whose expansion about the minimum should take a form $\;\mathrm{const.} + \lambda_\varphi\, (\varphi^\dagger\varphi - v_{\mathrm{ew}}^2)^2 + \cdots$.

It is appropriate to regard the conditions discussed above as those to be imposed on the Wilson coefficients in the effective potential (in Landau gauge) *renormalized at the cut-off scale* $\mu = \Lambda$. Recall that, as we discussed in section 3, we may relate the charged lepton spectrum directly to the vacuum configuration of the effective potential at $\mu = \Lambda$.

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The advantage of choosing $\mu = \Lambda$ is that certain fine tuning can be avoided in this way. Let us describe how it works.

According to the argument above, $\varepsilon_{\Phi 3}$ is required to be much smaller than ε_K or ε_X in order to suppress corrections to Koide's formula. Furthermore, in section 4 we have seen that $\varepsilon_{\Phi 2}$ should be much smaller than $\varepsilon_{\Phi 3}$ to generate a realistic charged lepton spectrum. Hence, $\varepsilon_{\Phi 2}/\varepsilon_K$ and $\varepsilon_{\Phi 2}/\varepsilon_X$ should be quite small, of order 10^{-5} or less. On the other hand, the 1-loop correction by family gauge interaction to the effective potential induces $V_{\Phi 2}$. This indicates that a natural size of $\varepsilon_{\Phi 2}$ is order $\alpha_F^2 \sim 10^{-3}$ or larger within the $\mathrm{U}(3) \times \mathrm{SU}(2)$ effective theory, assuming the relation (3.20). Thus, in order to realize $\langle \Phi \rangle \approx \Phi_0$, a fine tuning of $\varepsilon_{\Phi 2}$ seems to be requisite (provided magnitudes of ε_K and ε_X are moderate). This argument, however, does not apply at the cut-off scale $\mu = \Lambda$: Since $\mathrm{SU}(9) \times \mathrm{U}(1)$ symmetry forbids $V_{\Phi 2}$ and $V_{\Phi 3}$ in the theory above the scale Λ, both $\varepsilon_{\Phi 2}$ and $\varepsilon_{\Phi 3}$ are expected to be suppressed at $\mu = \Lambda$ in the $\mathrm{U}(3) \times \mathrm{SU}(2)$ theory. They are determined by the matching conditions at $\mu = \Lambda$. Radiative corrections within the $\mathrm{U}(3) \times \mathrm{SU}(2)$ effective theory essentially do not exist at this scale.

Another advantage of choosing $\mu = \Lambda$ in the effective potential is that $\mathrm{SU}(9) \times \mathrm{U}(1)$ symmetry breaking effects on $\langle X \rangle = X_0$ are also expected to be suppressed. In other words, the wave function renormalizations are common to X_A, X_S^1 and X_S^5, to a good approximation. It helps to keep the first condition of eq. (4.2) precise, which follows from eqs. (5.5) and (5.6).

The sizes of Wilson coefficients of operators non-invariant under $\mathrm{SU}(9)\times\mathrm{U}(1)$ at $\mu = \Lambda$ depend on the dynamics how the breakdown of $\mathrm{SU}(9) \times \mathrm{U}(1)$ gauge symmetry occurs in the theory above the scale Λ. For example, one can imagine cases in which these operators are proportional to (powers of) a VEV of some scalar field which breaks $\mathrm{SU}(9) \times \mathrm{U}(1)$ symmetry. Then, an operator whose dimension is n would have a coefficient of order $g\,\Lambda^k/M^{n+k-4}$, where g is a combination of coupling constants, Λ is a typical scale of the scalar VEV, k is the power of the VEV, and M represents an $\mathrm{SU}(9)\times\mathrm{U}(1)$-invariant mass scale much larger than Λ. There are no evident conflicts between this naive estimate and the conditions on the Wilson coefficients which we derived above, presuming that g can be small but cannot be much larger than unity. For instance, applying the estimate to the parameters of $V(\Phi, X)$ and $V_{\Phi 2}$, we find

$$\mathrm{SU}(9) \times \mathrm{U}(1) \text{ invariant}: \qquad \lambda, \varepsilon_{K1} \lesssim \mathcal{O}(1), \qquad \lambda v^2 \lesssim M; \tag{5.21}$$

$$\mathrm{SU}(9) \times \mathrm{U}(1) \text{ non-invariant}: \quad \varepsilon_{X1} \lesssim \frac{f_X^2 \Lambda^2}{v^4}, \quad \varepsilon_{\Phi 2}, \varepsilon_{\Phi 3} \lesssim \frac{\Lambda^k}{M^k}, \quad \varepsilon_{\Phi X1} \lesssim \frac{f_X^2}{v^2}\frac{\Lambda^k}{M^k}, \tag{5.22}$$

which are compatible with the desired hierarchy of the parameters $\varepsilon_{\Phi 2} \ll \varepsilon_{\Phi 3}, \varepsilon_{\Phi X1} \ll \varepsilon_{K1}, \varepsilon_{X1}$. Of course, one should keep in mind that the above estimates are heavily dependent on the dynamics above the cut-off scale.

We may speculate on a possible scenario above the cut-off scale which may lead to (part of) the desirable hierarchical relations. Suppose that the symmetry breaking $\mathrm{SU}(9)\times \mathrm{U}(1) \to \mathrm{U}(3) \times \mathrm{SU}(2)$ is induced by a condensate of a scalar field $T^{\alpha\beta}_{\rho\sigma}$, which is a 4th-rank tensor under $\mathrm{SU}(9)$. Indeed if $\langle T^{\alpha\beta}_{\rho\sigma} \rangle \sim \mathrm{tr}(T^\alpha T^{\beta *} T^{\rho *} T^\sigma)$, this symmetry breaking takes place. Through the first diagram shown in figure 6, the operator $\varepsilon_{\Phi 3}\,\mathrm{tr}(\Phi\,\Phi^T\,\Phi^*\,\Phi^\dagger)$ may be

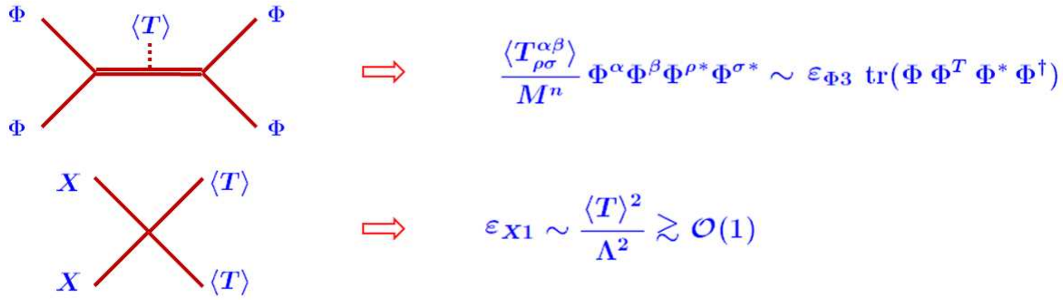

Figure 6. Speculation on underlying physics that may generate SU(9) $\times$ U(1) non-invariant operators.

induced; the double line denotes a heavy degree of freedom with an SU(9) $\times$ U(1)-invariant mass scale M. Since $\langle T^{\alpha\beta}_{\rho\sigma}\rangle \sim \mathcal{O}(\Lambda)$, the coefficient $\varepsilon_{\Phi 3} \sim \Lambda/M$ would be a small parameter provided $M \gg \Lambda$. $\varepsilon_{\Phi 2}$ is even more suppressed, since the operator $\varepsilon_{\Phi 2}\,\mathrm{tr}(\Phi^\dagger\,\Phi\,\Phi^\dagger\,\Phi)$ cannot be generated by a single insertion of $\langle T^{\alpha\beta}_{\rho\sigma}\rangle$ at tree level. Either two insertions of $\langle T^{\alpha\beta}_{\rho\sigma}\rangle$ or a loop correction is necessary, which leads to additional suppression factors. The second diagram in figure 6 would induce the operator $\varepsilon_{X1}\,\mathrm{tr}(T^\alpha\,T^\rho\,T^\beta\,T^\sigma)\,X^{\alpha\beta}\,X^{\rho\sigma *}$ (together with other operators). Since there is no intermediate heavy degree of freedom, the induced coupling ε_{X1}, when normalized by Λ, would be order 1. In order to generate $\varepsilon_{\Phi X1}$ with a desired order of magnitude, we need to suppose a more complicated scenario, but we do not pursue this further here, since anyway the argument is quite hand-waving, without any explicit model above the cut-off scale.

To end this section, let us comment on the fine tuning problem in maintaining a large hierarchy between the scales, which we mentioned in section 1. In the derivation of the potential of φ, it appears unnatural that $v_{\mathrm{ew}}(\ll v)$ determines the scale, since the natural scales involved in the effective potential are Λ and v before substituting $\Phi \approx \Phi_0$ and $X \approx X_0$. Currently we do not have any reasonable idea on how this hierarchy problem may be resolved.

6 Inclusion of another scalar field

Our goal is to generate the spectrum of the charged leptons (m_e, m_μ, m_τ) such that it satisfies Koide's formula with a high accuracy and is proportional to (v_1^2, v_2^2, v_3^2) approximately, where v_i's are given in eq. (4.12). For this purpose, we need to introduce yet another scalar field. This is because, if we construct the higher-dimensional operator $\mathcal{O}^{(\ell)}$ only from the fields ψ_L, e_R, Φ, X and φ, the corresponding charged lepton mass matrix cannot be brought to a diagonal form with any choice of basis allowed by U(3) $\times$ SU(2) gauge symmetry. (Note that Φ_0 is not diagonal.) The radiative corrections discussed in section 3 will be altered if the mass matrix cannot be brought to a diagonal form, and the QED correction will not be canceled.

Thus, we introduce a (dimensionless) scalar field Σ_Y which is in the $(\mathbf{6}, 1, Q_Y)$ under SU(3) $\times$ SU(2) $\times$ U(1). It is given as a 3-by-3 symmetric matrix and transforms as $\Sigma_Y \to$

$U\Sigma_Y U^T$. Consider the potentials

$$V_{\Sigma_Y} = -\varepsilon_{Y1}\, v^4\, \mathrm{tr}\left(\Sigma_Y^\dagger \Sigma_Y\right) + \varepsilon_{Y2}\, v^4\, \mathrm{tr}\left(\Sigma_Y^\dagger \Sigma_Y \Sigma_Y^\dagger \Sigma_Y\right) + \varepsilon_{Y3}\, v^4 \left[\mathrm{tr}\left(\Sigma_Y^\dagger \Sigma_Y\right)\right]^2 , \tag{6.1}$$

$$V_{\Phi\Sigma_Y} = -\varepsilon_{\Phi Y1}\, \mathrm{tr}\left(\Sigma_Y^\dagger \Phi\Phi^\dagger\, \Sigma_Y \Phi^* \Phi^T\right) . \tag{6.2}$$

We take all the parameters ε_{Y1}, ε_{Y2}, ε_{Y3}, $\varepsilon_{\Phi Y1}$ to be positive. One can show that, for a given Φ and in the limit $\varepsilon_{\Phi Y1} \ll \varepsilon_{Y1}, \varepsilon_{Y2}, \varepsilon_{Y3}$, $V_{\Sigma_Y} + V_{\Phi\Sigma_Y}$ is minimized at

$$\Sigma_Y = \sigma\, U_\Phi\, U_\Phi^T \quad ; \quad \sigma = \sqrt{\frac{\varepsilon_{Y1}}{2\,(\varepsilon_{Y2} + 3\,\varepsilon_{Y3})}} . \tag{6.3}$$

Here, U_Φ is a unitary matrix which diagonalizes $\Phi\Phi^\dagger$, i.e., $U_\Phi^\dagger \Phi\Phi^\dagger U_\Phi$ is a diagonal matrix; see appendix D.1. In the case that $\Phi = \Phi_0$, the corresponding unitary matrix is given by

$$U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i \\ -i & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad ; \quad U_0^\dagger\, \Phi_0\, U_0 = \Phi_d = \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix} . \tag{6.4}$$

Therefore, we may incorporate Σ_Y in the operator $\mathcal{O}^{(\ell)}$ to diagonalize the lepton mass matrix.

As in the previous section, we embed Σ_Y in a representation of a larger symmetry group, which is valid above the cut-off scale Λ. We could find a reasonable potential only when we embed Σ_Y to a second-rank antisymmetric representation, and this is not possible with SU(9) $\times$ U(1). We find a way out by enlarging the gauge group. Instead of SU(9) $\times$ U(1) we assume that SU(nm) $\times$ U(1) ($n \geq 4$, $m \geq 5$) gauge symmetry is exact above the cut-off scale.[14] Under this symmetry group, Φ is embedded in the $(\boldsymbol{nm}, 1)$. We denote the field in the latter representation by $\overline{\Phi}^\xi$ ($0 \leq \xi \leq nm-1$) and identify $\overline{\Phi}^\xi = \Phi^\xi$ for $0 \leq \xi \leq 8$. $\overline{\Phi}$ decomposes into a $(\mathbf{3}, \mathbf{3}, 1)$ ($= \Phi$), $n-3$ $(\mathbf{1}, \mathbf{3}, 1)$'s, $m-3$ $(\mathbf{3}, \mathbf{1}, 1)$'s, and $(n-3)(m-3)$ singlets after the symmetry is broken down to SU(3) $\times$ SU(2) $\times$ U(1). Similarly X is embedded in the second-rank symmetric representation of SU(nm), denoted by $\overline{X}$, and $\overline{X}^{\xi\eta} = X^{\xi\eta}$ for $0 \leq \xi, \eta \leq 8$. Σ_Y is embedded in the second-rank antisymmetric representation of SU(nm), denoted by $\overline{Y}$; see appendix D.2 for the explicit relation between $\overline{Y}$ and Σ_Y. Both $\overline{X}$ and $\overline{Y}$ are unitary fields. The kinetic terms of $\overline{X}$ and $\overline{Y}$ are normalized as $f_{\overline{X}}^2 |(D_\mu \overline{X})^{\xi\eta}|^2$ and $f_{\overline{Y}}^2 |(D_\mu \overline{Y})^{\xi\eta}|^2$, respectively, where $f_{\overline{X}}$ and $f_{\overline{Y}}$ are assumed to be much smaller than v.

We examine the general potential of $\overline{\Phi}$, $\overline{X}$ and $\overline{Y}$ which is invariant under SU(nm) $\times$ U(1). In particular, we would like to see if the potential can be minimized at

$$\overline{\Phi}^\xi = \begin{cases} \Phi_0^\xi & (0 \leq \xi \leq 8) \\ 0 & (\xi > 8) \end{cases} , \tag{6.5}$$

$$\overline{X}^{\xi\eta} = -2\, \delta^{\xi 0}\, \delta^{\eta 0} + \delta^{\xi\eta} , \tag{6.6}$$

$$\Sigma_Y = \sigma\, U_0 U_0^T , \tag{6.7}$$

[14] SU(nm) includes SU(n) $\times$ SU(m) as a maximal subgroup. Below the cut-off Λ, the symmetry is broken down to SU(3) $\times$ SU(2) $\times$ U(1), where SU(3) and SU(2), respectively, are subgroups of SU(n) and SU(m): SU(3) is embedded trivially in SU(n), i.e., the $\boldsymbol{n}$ decomposes into a $\mathbf{3}$ and $n-3$ singlets; SU(2) is a maximal subgroup of SU(3)$'$, which is embedded in SU(m) trivially.

without fine tuning of parameters, where Σ_Y is embedded in $\overline{Y}$ appropriately. The general potential can be written in the following form:[15]

$$V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\overline{\Phi}\,\overline{X}\,\overline{Y}} = \sum_{\substack{p_i,p'_i,q_i,q'_i\geq 0\\ Q_{\mathrm{tot}}=0}} C(p_i,p'_i,q_i,q'_i) \prod_{i=1}^{4} z_i(p_i)^{q_i}\,\{z_i(p'_i)^{q'_i}\}^*\,. \tag{6.8}$$

$z_i(p_i)$ denote SU(nm)-invariant operators[16]

$$z_1(p_1) = \mathrm{Tr}\Big[\big(\overline{X}^\dagger\cdot\overline{Y}\big)^{2p_1}\Big], \qquad z_2(p_2) = \overline{\Phi}\cdot\big(\overline{X}^\dagger\cdot\overline{Y}\big)^{p_2}\cdot\overline{\Phi}^*, \tag{6.9}$$

$$z_3(p_3) = \overline{\Phi}\cdot\big(\overline{X}^\dagger\cdot\overline{Y}\big)^{2p_3}\cdot\overline{X}^\dagger\cdot\overline{\Phi}\,, \qquad z_4(p_4) = \overline{\Phi}\cdot\big(\overline{Y}^\dagger\cdot\overline{X}\big)^{2p_4+1}\cdot\overline{Y}^\dagger\cdot\overline{\Phi}\,. \tag{6.10}$$

The summation is constrained to the sector with vanishing U(1) charge by the condition

$$Q_{\mathrm{tot}} \equiv q_i \sum_i Q(z_i(p_i)) - q'_i \sum_i Q(z_i(p'_i)) = 0\,, \tag{6.11}$$

where $Q(z)$ represents the U(1) charge of the operator z. Due to complexity of the potential, we were unable to clarify if the configuration eqs. (6.5)–(6.7) can be a classical vacuum in a sufficiently general region of the parameter space spanned by $\{C(p_i,p'_i,q_i,q'_i)\}$. We only confirmed this in a restricted region of the parameter space: For definiteness, we set $(n,m) = (4,5)$; we consider the parameter space spanned by $C(p_i,p'_i,q_i,q'_i)$ for $p_i,p'_i \leq 1$ and arbitrary q_i,q'_i, while all other $C(p_i,p'_i,q_i,q'_i)$ are set equal to zero. In this restricted parameter space, there exists a domain with a finite volume (non-zero measure), in which $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\overline{\Phi}\,\overline{X}\,\overline{Y}}$ is minimized at the configuration eqs. (6.5)–(6.7) by appropriately choosing $\overline{Y}$. Namely, the desired configuration is a vacuum (in fact, one of many degenerate vacua) in this domain. See appendix D.3 for details. This feature may indicate that the configuration eqs. (6.5)–(6.7) can be a vacuum of $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\overline{\Phi}\,\overline{X}\,\overline{Y}}$ without fine tuning of the parameters in the potential.

Operators non-invariant under SU$(nm)\times$U(1) are induced at $\mu \leq \Lambda$. Suppose the following operators are induced:

$$V_{\overline{\Phi},\mathrm{resid}} = \varepsilon_{\overline{\Phi}}\, v^2 \sum_{\xi\geq 9} \big|\overline{\Phi}^\xi\big|^2\,, \tag{6.12}$$

$$V_{X1},\ V_{\Phi 3},\ V_{\Phi X1} \quad \text{as defined in eqs. (5.8), (4.7), (5.10)}\,, \tag{6.13}$$

$$V_{\Sigma_Y},\ V_{\Phi\Sigma_Y} \quad \text{as defined in eqs. (6.1), (6.2)}\,. \tag{6.14}$$

Then, with appropriate hierarchy of the parameters, which we already discussed in this and previous sections, we have the configuration eqs. (6.5)–(6.7) as a global minimum

[15] For instance, the right-hand side of eq. (5.20), after replacing Φ by $\overline{\Phi}$ and X by $\overline{X}$, is included in this expression; it corresponds to the terms for which $q_1, q'_1, p_2, p'_2, p_3, p'_3, q_4, q'_4 = 0$.

[16] The dot $(\cdot)$ denotes contraction of SU(nm) indices $\xi,\eta,\ldots$; $\overline{X}^p = \underbrace{\overline{X}\cdot\overline{X}\cdots\overline{X}}_{p}$, $\mathrm{Tr}(\overline{X}) = \overline{X}^{\xi\xi}$, etc.

of the potential.[17] Generally, it depends on the dynamics above the cut-off scale which $\mathrm{SU}(nm) \times \mathrm{U}(1)$-breaking operators are induced, and a set of operators more general than eqs. (6.12)–(6.14) can also lead to the same vacuum configuration; see the discussion in the previous section.

For later convenience, we may take the potential

$$V(\overline{\Phi}, \overline{X}, \overline{Y}) = V_{\overline{\Phi}1} + V_{\overline{K}1} + V_{\overline{\Phi},\mathrm{resid}} + V_{X1} + V_{\Phi 3} + V_{\Phi X1} + V_{\Sigma_Y} + V_{\Phi\Sigma_Y} \tag{6.15}$$

with

$$V_{\overline{\Phi}1} = \lambda \left(\frac{1}{2}\, \overline{\Phi}^{\xi} \overline{\Phi}^{\xi *} - v^2 \right)^2 , \tag{6.16}$$

$$V_{\overline{K}1} = \varepsilon_{K1} \left| \overline{\Phi}^{\xi}\, \overline{X}^{\xi\eta *}\, \overline{\Phi}^{\eta} \right|^2 , \tag{6.17}$$

as a reference potential, instead of $V(\Phi, X)$ defined in eq. (5.7). For definiteness, we set $(n, m) = (4, 5)$. In this case, we obtain the desired vacuum configuration, eqs. (6.5)–(6.7), in the limit $\varepsilon_{\Phi 3}, \varepsilon_{\Phi X1} \ll \varepsilon_{K1}, \varepsilon_{X1}$ and $\varepsilon_{\Phi Y1} \ll \varepsilon_{Y1}, \varepsilon_{Y2}, \varepsilon_{Y3}$, and with the additional conditions eqs. (5.12) and (D.13).

7 Higher-dimensional operator $\mathcal{O}^{(\ell)}$

We present a candidate of the higher-dimensional operator $\mathcal{O}^{(\ell)}$ which generates the charged lepton spectrum. The VEV Φ_0, given by eq. (4.8), cannot be brought to a diagonal form (in 3-by-3 matrix representation) using the $\mathrm{U}(3) \times \mathrm{SU}(2)$ transformation. Hence, the operator such as the one in eq. (3.6) is inappropriate. In terms of the fields which we introduced, a simplest possibility may be given by

$$\mathcal{O}_3^{(\ell)} = \frac{\kappa^{(\ell)}(\mu)}{\Lambda^2}\, \bar{\psi}_L\, \Phi\, X_A^T\, \Phi\, X_A^T\, \Sigma_Y\, \varphi\, e_R\, . \tag{7.1}$$

In this case $2Q_X + Q_Y = 0$ is required, such that this operator becomes U(1)-invariant. Since $\langle X_A \rangle \approx -\frac{5}{3}\mathbf{1}$ in the basis where $\langle \Phi \rangle \approx \Phi_0$ [see eq. (B.7) in appendix B], the above operator can be approximately rendered to the form of eq. (3.4) by the change of basis, $\psi_L \to U_0\, \psi_L$ and $e_R \to U_0^*\, e_R$. The corresponding charged lepton mass matrix reads

$$\mathcal{M}_\ell = \frac{25\, \kappa^{(\ell)}(\mu)\, \sigma\, v_{\mathrm{ew}}}{9\sqrt{2}\, \Lambda^2} \begin{pmatrix} v_1(\mu)^2 & 0 & 0 \\ 0 & v_2(\mu)^2 & 0 \\ 0 & 0 & v_3(\mu)^2 \end{pmatrix} , \tag{7.2}$$

up to corrections of $\mathcal{O}(\varepsilon_\Phi/\varepsilon_K)$, $\mathcal{O}(\varepsilon_\Phi/\varepsilon_X)$, $\mathcal{O}(\varepsilon_{\Phi 2}/\varepsilon_{\Phi 3})$ or $\mathcal{O}(\varepsilon_{\Phi Y1}/\varepsilon_{Yi})$, where ε_Φ represents $\varepsilon_{\Phi X1}$, $\varepsilon_{\Phi 2}$, $\varepsilon_{\Phi 3}$, etc.

We should check whether the radiative correction induced by exchange of Φ (figure 7) violates Koide's mass formula or not. With the reference potential eq. (6.15), we consider

[17] There are a number of unwanted massless modes at this minimum. They are included in $\overline{X}$, $\overline{Y}$ and do not couple directly to ψ_L, e_R, φ, Φ and Σ_Y. Although it is straightforward to write down $\mathrm{U}(3) \times \mathrm{SU}(2)$-invariant operators which give masses to these massless modes, we do not include those operators, for the sake of simplicity. In particular, they do not affect the formulas given in the following discussion.

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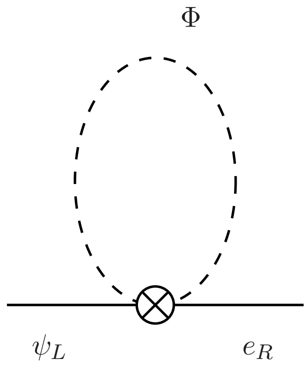

Figure 7. Scalar loop diagram which contributes to charged lepton masses. Dashed line represents all the mass eigenstates in Φ which couple to the operator $\mathcal{O}_3^{(\ell)}$ (represented by $\otimes$).

the limit $\varepsilon_{X1} \to \infty$ and $\varepsilon_{\Phi 3}, \varepsilon_{\Phi X1}, \varepsilon_{\Phi Y1} \to 0$ consistently with the assumed hierarchy of the parameters. Physical modes of X_A decouple in the former limit. Thus, we consider only $V_{\overline{\Phi}1}$ and $V_{\overline{K}1}$. One may determine the scalar mass eigenstates explicitly and find

$$\delta_\Phi m_i^{\rm pole} = -\frac{2\,\lambda}{(4\pi)^2}\left[\log\left(\frac{\mu^2}{4\lambda v_0^2}\right)+1\right] m_i(\mu)\,. \tag{7.3}$$

Since it has a form const. $\times\, m_i$, Koide's formula will not be affected. This result may be more non-trivial than one might think at a first glance, since the diagram in figure 7 corresponds to incorporating the class of (infinite number of) 1-loop diagrams shown in figure 2. As a cross check, we also computed the coefficient of $\log \mu^2$ through renormalization of the operator $\mathcal{O}_3^{(\ell)}$ in the symmetric phase ($v_0 = 0$).

There are only three physical modes of Φ which gain masses of order v_0; these are $\delta\Phi^\alpha \equiv \Phi^\alpha - \Phi_0^\alpha$ which are proportional to Φ_0^α, $X_0^{\alpha\beta}\,\Phi_0^\beta$ and $i\,X_0^{\alpha\beta}\,\Phi_0^\beta$. The first mode gives the correction eq. (7.3), while the contributions of the second and third modes cancel. Other modes have masses suppressed by $\varepsilon_{\Phi 3}, \varepsilon_{\Phi X1}, \varepsilon_{\Phi Y1}$, so that their contributions to the loop diagram in figure 7 are suppressed.

In fact the same features apply to the general potential of Φ and X, if the parameters of the potential satisfies the hierarchical relations required to realize the vacuum configuration $\Phi = \Phi_0$ and $X = X_0$ (as discussed in sections 5 and 6). Namely, absence of radiative corrections to Koide's formula induced by scalar exchanges can be shown in the limit $\varepsilon_{\Phi 3}, \varepsilon_{\Phi X1}, \varepsilon_{\Phi Y1}, \text{etc.} \to 0$ (assuming that contributions from physical modes of X decouple also in this case).

As we discussed in section 2 with an example of underlying mechanism, it is assumed that operators other than $\mathcal{O}_3^{(\ell)}$, which contribute to the charged lepton masses at higher orders of $1/\Lambda$, are absent (or strongly suppressed) at $\mu = \Lambda$. Since these operators are non-invariant under $\mathrm{SU}(nm)\times\mathrm{U}(1)$, sizes of these operators are determined by the physics above the scale Λ. Within the $\mathrm{U}(3)\times\mathrm{SU}(2)$ effective theory starting from this boundary condition, other operators are not induced radiatively at lower energy scales and the relation (3.21) is preserved (see also the discussion in section 3).[18]

[18] A simpler operator such as $\bar{\psi}_L\,\Phi\,\Phi^\dagger\Sigma_Y\,\varphi\,e_R$ would be inappropriate for a candidate of $\mathcal{O}^{(\ell)}$, even though

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In the limit $\varepsilon_{\Phi 3}, \varepsilon_{\Phi X1} \ll \varepsilon_{K1}, \varepsilon_{X1}$ and $\varepsilon_{\Phi Y1} \ll \varepsilon_{Y1}, \varepsilon_{Y2}, \varepsilon_{Y3}$, the root-mass-ratios of the charged leptons are given by $\sqrt{m_i/m_0} = v_i/v_0$, where $m_0 = m_1 + m_2 + m_3$. They are in reasonable agreement with the corresponding experimental values as we have seen in eqs. (4.12) and (4.13). It would be instructive to see how much corrections are induced to these values by the small parameters in the potential. For simplicity, let us compute $\mathcal{O}(\varepsilon_\Phi)$ corrections to the charged lepton spectrum (m_1, m_2, m_3), corresponding to the potential eq. (6.15) and the higher-dimensional operator eq. (7.1). The $\mathcal{O}(\varepsilon_\Phi)$ corrections read

$$\delta\left(\sqrt{\frac{m_1}{m_0}}\right) \approx \left(-\frac{0.00209}{\varepsilon_{K1}} - \frac{0.00756}{\varepsilon_{X1}}\right)\varepsilon_{\Phi 3} + \left(\frac{0.0312}{\varepsilon_{K1}} + \frac{0.152}{\varepsilon_{X1}}\right)\varepsilon_{\Phi X1}\,, \tag{7.4}$$

$$\delta\left(\sqrt{\frac{m_2}{m_0}}\right) \approx \left(-\frac{0.00152}{\varepsilon_{K1}} - \frac{0.00406}{\varepsilon_{X1}}\right)\varepsilon_{\Phi 3} + \left(\frac{0.0227}{\varepsilon_{K1}} + \frac{0.0833}{\varepsilon_{X1}}\right)\varepsilon_{\Phi X1}\,, \tag{7.5}$$

$$\delta\left(\sqrt{\frac{m_3}{m_0}}\right) \approx \left(\frac{0.000406}{\varepsilon_{K1}} + \frac{0.00112}{\varepsilon_{X1}}\right)\varepsilon_{\Phi 3} + \left(-\frac{0.00607}{\varepsilon_{K1}} - \frac{0.0229}{\varepsilon_{X1}}\right)\varepsilon_{\Phi X1}\,. \tag{7.6}$$

As can be seen, the magnitude of the correction is larger for smaller mass eigenvalues, reflecting the nature of a hierarchical spectrum, as we discussed in section 4. Comparing to eqs. (4.12) and (4.13), one finds constraints on typical orders of magnitude of the parameters as $\varepsilon_{\Phi 3}/\varepsilon_{K1} \lesssim 10^0$, $\varepsilon_{\Phi 3}/\varepsilon_{X1} \lesssim 10^{-1}$, $\varepsilon_{\Phi X1}/\varepsilon_{K1} \lesssim 10^{-1}$, $\varepsilon_{\Phi X1}/\varepsilon_{X1} \lesssim 10^{-2}$, provided there is no correlation or fine tuning among these parameters, or with $\mathcal{O}(\varepsilon_{\Phi Y1}/\varepsilon_{Yi})$ corrections. On the other hand, the overall normalization,

$$m_1 + m_2 + m_3 = \frac{25\,\kappa^{(\ell)}(\mu)\,\sigma\,v_{\rm ew}}{9\sqrt{2}\,\Lambda^2}\,v_0^2\,\left[1 + \text{corr.}\right]\,, \tag{7.7}$$

is subject to radiative corrections induced by electroweak gauge interaction (including QED), family gauge interaction and scalar exchanges, in addition to the $\mathcal{O}(\varepsilon_\Phi)$ and $\mathcal{O}(\varepsilon_{\Phi Y1}/\varepsilon_{Yi})$ corrections.

We define the following quantity as a measure of the degree of violation of Koide's mass relation:

$$\Delta \equiv \frac{2(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2}{3\,(m_1 + m_2 + m_3)} - 1\,. \tag{7.8}$$

This quantity vanishes if Koide's relation is satisfied. With the reference potential and $\mathcal{O}_3^{(\ell)}$, the $\mathcal{O}(\varepsilon_\Phi)$ correction reads

$$\Delta = \left(\frac{1}{8\,\varepsilon_{K1}} + \frac{33 - 15\sqrt{2}\,x_0}{65\,\varepsilon_{X1}}\right)\bar{\varepsilon}_\Phi + \frac{67 - 75\sqrt{2}\,x_0}{390\,\varepsilon_{X1}}\,\varepsilon_{\Phi X1} \tag{7.9}$$

$$\approx \left(-\frac{0.00523}{\varepsilon_{K1}} - \frac{0.0171}{\varepsilon_{X1}}\right)\varepsilon_{\Phi 3} + \left(\frac{0.0781}{\varepsilon_{K1}} + \frac{0.346}{\varepsilon_{X1}}\right)\varepsilon_{\Phi X1}\,. \tag{7.10}$$

Comparing to the present experimental value $\Delta^{\rm exp} = (1.1 \pm 1.4) \times 10^{-5}$, we obtain constraints on typical sizes of the parameters more stringent than the previous ones: $\varepsilon_{\Phi 3}/\varepsilon_{K1} \lesssim 10^{-3}$, $\varepsilon_{\Phi 3}/\varepsilon_{X1} \lesssim 10^{-3}$, $\varepsilon_{\Phi X1}/\varepsilon_{K1} \lesssim 10^{-4}$, $\varepsilon_{\Phi X1}/\varepsilon_{X1} \lesssim 10^{-4}$.

it gives the desired spectrum at tree level: This operator induces a mass matrix $\delta\mathcal{M}_\ell \propto \mathbf{1}$ radiatively, upon contraction of Φ and $\Phi^\dagger$.

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It is easy to adjust the root-mass-ratios $\sqrt{m_i/m_0}$ to be consistent with the current experimental values without violating Koide's relation, as we discussed in section 4. For instance, it is achieved by incorporating $V_{\Phi 2}$ with $\varepsilon_{\Phi 2}/\varepsilon_{\Phi 3} \approx -6 \times 10^{-3}$ into the potential.

8 Relevant scales and further assumptions

Let us discuss the energy scales, Λ, $v_3(\sim v_0)$, f_X, involved in the present model. It would be unnatural if there is a large hierarchy between v_3 and Λ, or between f_X and v_3. As we speculated in section 3, the scale of U(3) symmetry breaking, typically given by v_3, may be at 10^2–10^3 TeV, such that the QED correction is cancelled within a scenario of unification of the electroweak $\mathrm{SU}(2)_L$ and family U(3) symmetries. There are two indications that the cut-off scale Λ and the U(3) symmetry breaking scale v_3 are not too far apart. One indication is the importance of the universality of the U(1) and SU(3) gauge coupling constants eq. (3.7). This universality may be protected above the cut-off scale by embedding SU(3) and U(1) in a simple group. If v_3 is very different from Λ, however, the values of the two coupling constants at $\mu \sim v_3$ would become too different. Another indication consists in the relation (7.2), from which one derives

$$\frac{v_3}{\Lambda} = \left(\frac{9\sqrt{2}\, m_\tau}{25\, \kappa^{(\ell)}\, \sigma\, v_{\mathrm{ew}}} \right)^{1/2} \approx \frac{1}{17\sqrt{\kappa^{(\ell)}\, \sigma}} \tag{8.1}$$

up to electroweak corrections, etc. σ is smaller than 1/2 and is expected to be not very much smaller. Although it depends on the mechanism how the higher-dimensional operator $\mathcal{O}^{(\ell)}$ is generated, if $\kappa^{(\ell)}$ is not large (as one naively expects), hierarchy between v_3 and Λ is mild. As for the scale f_X, it is required to be smaller than $\langle \Phi \rangle$ in order not to alter the spectrum of the family gauge bosons. Numerically $f_X \lesssim 0.3\, v_1 \sim 0.005\, v_3$ is required from the present constraint on Koide's formula.

There are a few more assumptions implicit in the present model, which we have not discussed so far. We assume that the SU(2) family gauge symmetry is broken spontaneously at a scale higher than $\langle \Phi \rangle$. This is required to protect the symmetry breaking pattern eq. (3.16), which constrains the form of the radiative correction by the U(3) gauge bosons. To achieve this, we need additional fields or dynamics, such as an SU(2) doublet scalar field whose VEV breaks SU(2). As yet, we have not succeeded to incorporate such a mechanism consistently into our model. Here, we simply assume that the breakdown of SU(2) has occurred without affecting the properties of our model described above.

We also assume cancellation of gauge anomalies and decoupling of unwanted fermions. Namely, we assume cancellation of anomalies introduced by the couplings of fermions to family gauge bosons, at the scale where $\mathrm{U}(3) \times \mathrm{SU}(2)$ symmetry is unbroken. This means that we need fermions other than the SM fermions. Fermions other than the SM fermions are requisite in our model also because ψ_L and e_R are embedded into larger multiplets of $\mathrm{SU}(nm)$. At lower energy scales, $\mu \ll v$, all the additional fermions are assumed to aquire masses of order v or larger, so that they decouple from the SM sector at and below the electroweak scale. Only the SM fermions remain at these scales. Presently we do not have a model which fully explains these features.

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9 Lepton flavor violating processes and other predictions

A most characteristic prediction of the present model is the existence of lepton-flavor violating processes induced by the family gauge interaction. In the scenario, in which the U(3) family gauge symmetry and $\mathrm{SU}(2)_L$ weak gauge symmetry are unified at 10^2–10^3 TeV scale, the family gauge bosons have masses of the order of the unification scale.

As it is clear from eq. (3.15), flavor violating decays of a charged lepton with only charged leptons and/or photons in the final state, such as $\mu \to 3\,e$ or $\mu \to e\,\gamma$, are forbidden. Flavor violating leptonic decays which involve neutrinos, such as $\mu^- \to e^- \nu_e \bar{\nu}_\mu$, are allowed, but the present experimental sensitivities are very low. Presumably, the most sensitive process is $K_L \to e\,\mu$, although we need to make assumptions on the quark sector. For instance, assuming that the down-type quarks are in the same representation of U(3) as the charged leptons, and that the mass matrices of the charged leptons and down-type quarks are simultaneously diagonalized in an appropriate basis, this process is induced by an effective 4-Fermi interaction connecting the first and second generations:

$$\mathcal{L}_{4f}^{(1,2)} = \frac{1}{2\,(v_1^2+v_2^2)} \Big[\big(\bar{d}\,\gamma^\nu\gamma_5\, s + \bar{s}\,\gamma^\nu\gamma_5\, d\big)\,(\bar{e}\,\gamma_\nu\gamma_5\,\mu + \bar{\mu}\,\gamma_\nu\gamma_5\, e) \\ - \big(\bar{d}\,\gamma^\nu\, s - \bar{s}\,\gamma^\nu\, d\big)\,(\bar{e}\,\gamma_\nu\,\mu - \bar{\mu}\,\gamma_\nu\, e) \Big] + \cdots . \tag{9.1}$$

We find

$$\Gamma(K_L \to e\mu) \approx \frac{m_\mu^2 m_{K_L} f_K^2}{16\pi v_2^4}\,. \tag{9.2}$$

Comparing to the present experimental bound $\mathrm{Br}(K_L \to e\mu) < 4.7 \times 10^{-12}$ [2], we obtain a limit $v_2 \gtrsim 5 \times 10^2$ TeV. Naively this limit may already be marginally in conflict with the estimated unification scale in the above scenario. We should note, however, that this depends rather heavily on our assumptions on the quark sector. In the case that there exist additional factors in the quark sector which suppress the decay width by a few orders of magnitude, we may expect a signal for $K_L \to e\mu$ not far beyond the present experimental reach. Similarly the process $K^+ \to \pi^+ e^- \mu^+$ may also be observable in the future.

Another interesting observation, although it is much more model dependent, is the following. In order to stabilize Koide's formula, in our model, it is necessary to suppress $\mathrm{SU}(nm)\times\mathrm{U}(1)$ non-invariant operators in the potential of Φ. This indicates that Φ includes physical modes which are much lighter than $v \sim 10^2$–10^3 TeV. In particular, the lightest one, being singlet under the SM gauge group, may decay into leptons through the family gauge interaction or the operator $\mathcal{O}^{(\ell)}$ with a significant branching ratio. Hence, if this lightest scalar boson happens to be produced at the LHC, an excess in multi-lepton final states may be observed.

10 Summary and discussion

In this paper, we propose a model of charged lepton sector, in the context of an EFT valid below the cut-off scale Λ, which predicts a charged lepton spectrum consistently with the experimental values. In particular, we implement specific mechanisms into the model, such that the spectrum satisfies Koide's mass formula within the present experimental accuracy. In this model radiative corrections as well as other corrections to Koide's formula are kept under control, and this feature primarily differentiates the present model from the other models in the literature which predict Koide's formula. By studying within EFT, we circumvent many problems, at the price of introducing the cut-off scale at 10^2–10^3 TeV scale, while non-trivial relations between family symmetries and observed charged lepton spectrum can still be investigated.

In our model, we adopt a mechanism, through which the charged lepton mass matrix becomes proportional to the square of the VEV of a scalar field Φ [10]. On the basis of this mechanism, we incorporate two new mechanisms in the model which are worth emphasizing:

(i) The radiative correction to Koide's formula induced by family gauge interaction has the same form as the QED correction with opposite sign. This form is determined by the symmetry breaking pattern eq. (3.16) and the representations of ψ_L and e_R. Within a unification scenario, cancellation of the QED correction can take place.

(ii) A charged lepton spectrum, which has a hierarchical structure and approximates the experimental values, follows from a simple potential $V_{\Phi 3}$, under the condition that Koide's formula is protected.

Existence of such simple mechanisms may indicate relevance of $\mathrm{U}(3) \times \mathrm{SU}(2)$ family gauge symmetry in relation to the charged lepton spectrum.

Our model is constructed as an effective theory valid below the cut-off scale Λ respecting this symmetry. We introduce scalar fields $\overline{\Phi}$, $\overline{X}$ and $\overline{Y}$ as multiplets of $\mathrm{SU}(nm) \times \mathrm{U}(1)$, in which $\mathrm{U}(3) \times \mathrm{SU}(2)$ is embedded. It is assumed that $\mathrm{SU}(nm) \times \mathrm{U}(1)$ is spontaneously broken to $\mathrm{U}(3) \times \mathrm{SU}(2)$ below the scale Λ. We minimize the potential of the scalar fields and determine its classical vacuum. The charged lepton masses are related to the VEVs of the scalar fields at scale $\mu = \Lambda$, $m_i^{\rm pole} \propto v_i(\Lambda)^2$; at this scale radiative corrections to the VEVs essentially vanish within the effective theory. Then, the mass matrix of the charged leptons are given in terms of the VEVs, such that Koide's mass formula is stabilized, and that the spectrum agrees with the experimental values. This is achieved formally without fine tuning of parameters in the model, except for (a) the tuning required for stabilization of the electroweak scale $v_{\rm ew}$, and (b) the tuning required for the cancellation of the QED correction, that is, realizing $\alpha_F = \frac{1}{4}\alpha$ at relevant scales. We argue that the latter tuning can be replaced by a tuning of the unification scale, within a scenario in which U(3) family gauge symmetry and $\mathrm{SU}(2)_L$ weak gauge symmetry are unified at 10^2–10^3 TeV scale.

In addition our model may contain following fine tuning. We were unable to explore the parameter space of the $\mathrm{SU}(nm) \times \mathrm{U}(1)$-invariant potential $V^{\mathrm{SU}(nm)\times \mathrm{U}(1)}_{\overline{\Phi}\,\overline{X}\,\overline{Y}}$ sufficiently, due to technical complexity. It may be the case that certain fine tuning is necessary to realize the configuration eqs. (6.5)–(6.7) as a classical vacuum.

Evidently the present model is incomplete, since it is restricted to the charged lepton sector. The model should be implemented in a larger framework which incorporates at least the following aspects missing in the present model: (i) Including the quarks and ex-

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plaining the masses and mixings of the quarks and neutrinos; (ii) Cancellation of anomalies introduced by the couplings of fermions to family gauge bosons; (iii) Unification of U(3) and $SU(2)_L$ gauge symmetries at 10^2–10^3 TeV scale. Possibly these problems are solved simultaneously in some model, and one anticipates that such a model would necessarily contain a large number of new particles, for the following reasons: (a) all the particles are embedded into multiplets of large groups, especially if one also requires to unify hypercharge U(1) and color SU(3) gauge groups together with U(3) and $SU(2)_L$; (b) additional fermions are necessary to cancel anomalies; and (c) scalar fields would be necessary to give masses of order $\langle\Phi\rangle$ to fermions (apart from the SM fermions) through their VEVs [19].

Although our model predicts a realistic lepton spectrum, in fact many of the questions are simply reassigned to physics above the cut-off scale and remain unanswered: While we replaced the conditions on the lepton spectrum by the boundary conditions of the effective potential, we do not address which dynamics leads to these boundary conditions. (Only a speculation is given.) We may nevertheless state that not only did we circumvent fine tuning but also the problems actually simplified. The required boundary conditions are certain hierarchical structure among the couplings of the effective potential. These conditions would be simpler to realize than, for instance, to realize Koide's relation among the lepton Yukawa couplings with 10^{-5} accuracy *a priori*.

Phenomenologically our model predicts existence of lepton violating processes at 10^2–10^3 TeV scale, assuming the unification scenario at this scale. The processes $K_L \to \mu e$ and $K^+ \to \pi^+ e^- \mu^+$ are expected to be sensitive to the predictions of our model, although we need additional assumptions on the quark sector. Stability of Koide's formula indicates existence of light modes in Φ, and the lightest mode may decay into leptons with a significant branching ratio; they may generate an interesting signal at the LHC.

It is unlikely that the present model describes Nature correctly to the details, since we can easily construct variants of the present model with similar complexity. Overall, the present model is rather complicated, and the source of complexity is conspiracy to realize Koide's formula with a high accuracy. Hence, we place more emphasis on the major mechanisms incorporated in the model, which look appealing and may reflect physics that governs the spectrum of the charged leptons.

Acknowledgments

The author is grateful to K. Tobe for discussion. This work is supported in part by Grant-in-Aid for scientific research No. 17540228 from MEXT, Japan.

A Generators of U(3)

The generators for the representation $(\mathbf{3}, 1)$ of $\mathrm{U}(3) \simeq \mathrm{SU}(3) \times \mathrm{U}(1)$ are given by

$$
\begin{aligned}
T^0 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad
T^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
T^2 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T^3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
T^4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad
T^5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
T^6 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad
T^7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad
T^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{aligned}
\tag{A.1}
$$

which satisfy eq. (3.2).

B Decomposition of X under U(3) × SU(2)

X, which is in the $(\mathbf{45}, Q_X)$ of $\mathrm{SU}(9) \times \mathrm{U}(1)$, decomposes into $X_S^1(\mathbf{6}, \mathbf{1}, Q_X) \oplus X_S^5(\mathbf{6}, \mathbf{5}, Q_X) \oplus X_A(\bar{\mathbf{3}}, \mathbf{3}, Q_X)$ under $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. Explicitly they can be constructed as follows:

$$
\tilde{X}_{ik;jl} = X^{\alpha\beta}\, T^\alpha_{ij}\, T^\beta_{kl} \qquad \overset{\text{equiv.}}{\Longleftrightarrow} \qquad X^{\alpha\beta} = 4\, \tilde{X}_{ik;jl}\, T^\alpha_{ji}\, T^\beta_{lk}\,, \tag{B.1}
$$

$$
\begin{aligned}
\begin{pmatrix} \tilde{X}_S \\ \tilde{X}_A \end{pmatrix}_{ik;jl} &= \frac{1}{4} \left(\tilde{X}_{ik;jl} \pm \tilde{X}_{ki;jl} \pm \tilde{X}_{ik;lj} + \tilde{X}_{ki;lj} \right) && \text{(B.2)} \\
&= \frac{1}{2} \left(\tilde{X}_{ik;jl} \pm \tilde{X}_{ki;jl} \right)\,, && \text{(B.3)}
\end{aligned}
$$

$$
\begin{aligned}
(X_A)_{mn} &= \epsilon_{mik}\, \epsilon_{njl}\, (\tilde{X}_A)_{ik;jl}\,, && \text{(B.4)} \\
(X_S^1)_{ik} &= (\tilde{X}_S)_{ik;mm}\,, && \text{(B.5)} \\
(X_S^5)_{ik;jl} &= (\tilde{X}_S)_{ik;jl} - \frac{1}{3}\, (\tilde{X}_S)_{ik;mm}\, \delta_{jl}\,. && \text{(B.6)}
\end{aligned}
$$

When $\langle X \rangle = X_0$, the corresponding VEVs of X_A, X_S^1 and X_S^5 are given, respectively, by

$$
\begin{aligned}
\langle X_A \rangle_{mn} &= -\frac{5}{3}\, \delta_{mn}\,, && \text{(B.7)} \\
\langle X_S^1 \rangle_{ik} &= \frac{1}{6}\, \delta_{ik}\,, && \text{(B.8)} \\
\langle X_S^5 \rangle_{ik;jl} &= \frac{1}{12} (\delta_{il}\delta_{kj} + \delta_{ij}\delta_{kl}) - \frac{1}{18} \delta_{ik}\delta_{jl}\,. && \text{(B.9)}
\end{aligned}
$$

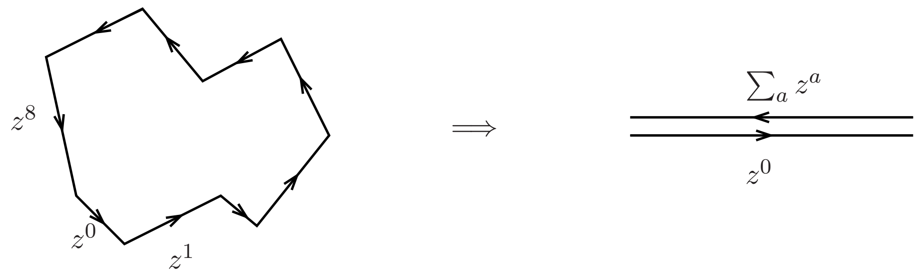

Figure 8. Eq. (C.3) corresponds to a nonagon with a fixed length of circumference in the complex plane. If we maximize $|z^0|$, the nonagon collapses to a line; after overall phase rotation, all z^α's can be made real, where $z^0 = v_0^2 > 0$ and $z^a \leq 0$.

C Properties of $V(\Phi, X)$

C.1 Maximizing $|\Phi^0|^2$

We impose the conditions

$$\Phi^{\alpha *}\Phi^\alpha = 2\, v_0^2 > 0\,, \qquad\qquad (\Phi^0)^2 = \Phi^a\, \Phi^a\,. \tag{C.1}$$

If we maximize $|\Phi^0|^2$ under these conditions, all Φ^α's can be made real simultaneously by a common phase rotation. Namely, there exists a phase θ such that $e^{-i\theta}\Phi^\alpha \in \mathbf{R}$ for all α.

Proof: let

$$z^0 = (\Phi^0)^2\,, \qquad z^a = -(\Phi^a)^2\,. \tag{C.2}$$

Then z^α's satisfy

$$\sum_{\alpha=0}^{8} z^\alpha = 0\,, \qquad \sum_{\alpha=0}^{8} |z^\alpha| = 2\, v_0^2 > 0\,. \tag{C.3}$$

These equations represent a nonagon with a fixed length of circumference in the complex plane. $|z^0| = |\Phi^0|^2$ is maximized when the nonagon collapses to a line, where all z^a's are parallel to one another and antiparallel to z^0 in the complex plane with

$$|z^0| = \sum_{a=1}^{8} |z^a| = v_0^2\,. \tag{C.4}$$

See figure 8.

C.2 Variation of $V_{\Phi 3}$ at $\Phi = \Phi_0$

The variation of $V_{\Phi 3}$, defined by eq. (4.7), is positive semi-definite at $\Phi = \Phi_0$ under $\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)$ transformation. Namely, if $\Phi_0' = U_1\Phi_0 U_2^\dagger$ with $U_1U_1^\dagger = U_2U_2^\dagger = \mathbf{1}$, $V_{\Phi 3}(\Phi_0') \geq V_{\Phi 3}(\Phi_0)$.

Proof: let

$$\Phi_1 \equiv U_2 U_1^\dagger \Phi_0' = U_2 \Phi_0 U_2^\dagger \,. \tag{C.5}$$

Then, noting

$$(\Phi_0^0)^2 = \Phi_0^a \, \Phi_0^a = v_0^2 \,, \tag{C.6}$$

Φ_1 satisfies the same relation:

$$(\Phi_1^0)^2 = \Phi_1^a \, \Phi_1^a = v_0^2 \,. \tag{C.7}$$

Since Φ_0^α's are real, Φ_1^α's are also real. According to section 4, Φ_0 is a configuration which minimizes $V_{\Phi 3}(\Phi)$ under the condition $(\Phi^0)^2 = \Phi^a \Phi^a = v_0^2$ and $\Phi^\alpha \in \mathbf{R}$. Therefore, $V_{\Phi 3}(\Phi_1) \geq V_{\Phi 3}(\Phi_0)$. Since Φ_1 and Φ_0' are connected by a $\mathrm{U}(3) \times \mathrm{SU}(2)$ transformation, $V_{\Phi 3}(\Phi_1) = V_{\Phi 3}(\Phi_0')$. It follows $V_{\Phi 3}(\Phi_0') \geq V_{\Phi 3}(\Phi_0)$.

Due to this property, $V_{X1} + V_K + V_{\Phi 3}$ is minimized at $X = X_0$ and $\Phi = \Phi_0$ under the constraint $\Phi^\alpha \in \mathbf{R}$ and $\Phi^\alpha \Phi^\alpha = 2\, v_0^2$.

C.3 CP transformations of Φ and X

CP transformation of Φ is defined by

$$(CP)\, \Phi(x)\, (CP)^\dagger = \Phi(\mathcal{P}x)^* \,, \tag{C.8}$$

where $\mathcal{P}x = (x^0, -\vec{x})$. Equivalently,

$$(CP)\, \Phi^\alpha(x)\, (CP)^\dagger = [\mathcal{C}^{\alpha\beta}\, \Phi^\beta(\mathcal{P}x)]^* \tag{C.9}$$

with

$$\mathcal{C}^{\alpha\beta} = [\mathrm{diag.}(+1, +1, -1, +1, +1, -1, +1, -1, +1)]_{\alpha\beta} \,. \tag{C.10}$$

Similarly CP transformation of X is defined by

$$(CP)\, X^{\alpha\beta}(x)\, (CP)^\dagger = [\mathcal{C}^{\alpha\alpha'}\, X^{\alpha'\beta'}(\mathcal{P}x)\, \mathcal{C}^{\beta'\beta}]^* \,, \tag{C.11}$$

or

$$\begin{aligned}
(CP)\, (X_A)_{ij}\, (CP)^\dagger \, &= \, (X_A)_{ij}^* \,, && \text{(C.12)}\\
(CP)\, (X_S^1)_{ij}\, (CP)^\dagger \, &= \, (X_S^1)_{ij}^* \,, && \text{(C.13)}\\
(CP)\, (X_S^5)_{ik;jl}\, (CP)^\dagger \, &= \, (X_S^5)_{ik;jl}^* \,. && \text{(C.14)}
\end{aligned}$$

CP transformations of other fields are the same as those of the SM.

For example, V_{X1} defined by eq. (5.8) is CP-invariant. An example of CP non-invariant operator is

$$V = \frac{1}{2i}(f - f^*) \tag{C.15}$$

with

$$f = [\epsilon_{ijk}\epsilon_{lmn}\, (X_A)_{il}\, (X_A)_{jm}\, (X_A)_{kn}][\epsilon_{i'j'k'}\epsilon_{l'm'n'}\, (X_S^1)_{i'l'}\, (X_S^1)_{j'm'}\, (X_S^1)_{k'n'}]^* \,. \tag{C.16}$$

C.4 Stability of V_X at $X = X_0$

When $V_X(X)$ is invariant under U(3) $\times$ SU(2) and CP, its first derivative vanishes $\partial V_X/\partial X^{\alpha\beta} = \partial V_X/\partial X^{\alpha\beta *} = 0$ at $X = X_0$.

Proof:

$$\delta V_X = \frac{\partial V_X}{\partial (X_A)_{ij}}(\delta X_A)_{ij} + \frac{\partial V_X}{\partial (X_A)^*_{ij}}(\delta X_A)^*_{ij} + \frac{\partial V_X}{\partial (X_S^1)_{ij}}(\delta X_S^1)_{ij} + \frac{\partial V_X}{\partial (X_S^1)^*_{ij}}(\delta X_S^1)^*_{ij} + \frac{\partial V_X}{\partial (X_S^5)_{ik;jl}}(\delta X_S^5)_{ik;jl} + \frac{\partial V_X}{\partial (X_S^5)^*_{ik;jl}}(\delta X_S^5)^*_{ik;jl}\,. \tag{C.17}$$

Due to the residual SU(2)$_V$ symmetry of the VEV $\langle X \rangle = X_0$, the differential coefficients evaluated at $X = X_0$ take following forms:

$$\left.\frac{\partial V_X}{\partial (X_A)_{ij}}, \frac{\partial V_X}{\partial (X_A)^*_{ij}}, \frac{\partial V_X}{\partial (X_S^1)_{ij}}, \frac{\partial V_X}{\partial (X_S^1)^*_{ij}}\right|_{X=X_0} \propto \delta_{ij}\,, \tag{C.18}$$

$$\left.\frac{\partial V_X}{\partial (X_S^5)_{ik;jl}}, \frac{\partial V_X}{\partial (X_S^5)^*_{ik;jl}}\right|_{X=X_0} = C_1\,\delta_{ik}\delta_{jl} + C_2\,\delta_{ij}\delta_{kl} + C_3\,\delta_{il}\delta_{jk}\,, \tag{C.19}$$

where C_i's are constants. Substituting to eq. (C.17), we have

$$\begin{aligned}\delta V_X\Big|_{X=X_0} &= C'_1\,(\delta X_A)_{ii} + C'_2\,(\delta X_A)^*_{ii} + C'_3\,(\delta X_S^1)_{ii} + C'_4\,(\delta X_S^1)^*_{ii} \\ &\quad + C'_5\,(\delta X_S^5)_{ik;ik} + C'_6\,(\delta X_S^5)^*_{ik;ik} + C'_7\,(\delta X_S^5)_{ik;ki} + C'_8\,(\delta X_S^5)^*_{ik;ki}\,,\end{aligned} \tag{C.20}$$

where we used $(\delta X_S^5)_{ii,jj} = 0$.

An arbitrary infinitesimal variation of X, which is symmetric and unitary, can be parametrized by

$$X^{\alpha\beta} + \delta X^{\alpha\beta} = W^{\alpha\alpha'}\,X^{\alpha'\beta'}\,W^{\beta\beta'} \tag{C.21}$$

with

$$W \simeq \begin{pmatrix} 1 + i\epsilon^R_{00} & i(\epsilon^R_{01} + i\epsilon^I_{01}) & \cdots & i(\epsilon^R_{08} + i\epsilon^I_{08}) \\ i(\epsilon^R_{01} - i\epsilon^I_{01}) & 1 + i\epsilon^R_{11} & \cdots & i(\epsilon^R_{18} + i\epsilon^I_{18}) \\ \vdots & \vdots & \ddots & \vdots \\ i(\epsilon^R_{08} - i\epsilon^I_{08}) & i(\epsilon^R_{18} - i\epsilon^I_{18}) & \cdots & 1 + i\epsilon^R_{88} \end{pmatrix}, \tag{C.22}$$

neglecting $\mathcal{O}(\epsilon^2)$ terms. An explicit calculation shows that, for a variation $X = X_0 + \delta X$, $(\delta X_A)_{ii}$, $(\delta X_S^1)_{ii}$, $(\delta X_S^5)_{ik;ik}$ and $(\delta X_S^5)_{ik;ki}$ depend only on $\epsilon^R_{\alpha\alpha}$ for $0 \le \alpha \le 8$. (In this proof, no sum is taken over α in $\epsilon^R_{\alpha\alpha}$ without explicit summation symbol $\sum_\alpha$.) Hence,

$$\delta V_X\Big|_{X=X_0} \simeq \sum_{\alpha=0}^{8} \epsilon^R_{\alpha\alpha}\,\frac{\partial}{\partial \epsilon^R_{\alpha\alpha}} V_X(X_0 + \delta X)\Big|_{\epsilon^R_{\alpha\beta}, \epsilon^I_{\alpha\beta}=0}\,. \tag{C.23}$$

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On the other hand, applying CP transformation eq. (C.11) to $X = X_0 + \delta X$, one finds that X_0 is CP-even, whereas all the coefficients of $\epsilon^R_{\alpha\alpha}$ in δX are CP-odd. This means, if V_X is CP-invariant,

$$\frac{\partial}{\partial \epsilon^R_{\alpha\alpha}} V_X(X_0 + \delta X)\bigg|_{\epsilon^R_{\alpha\beta},\epsilon^I_{\alpha\beta}=0} = 0 \qquad \text{for} \;\; 0 \le \alpha \le 8\,, \tag{C.24}$$

so that the first derivative vanishes, $\delta V_X|_{X=X_0} = 0$.

D $\overline{Y}$, Σ_Y and their potential

D.1 Minimum of $V_{\Sigma_Y} + V_{\Phi\Sigma_Y}$

We show that $V_{\Sigma_Y} + V_{\Phi\Sigma_Y}$, given by eqs. (6.1) and (6.2), is minimized at the configuration eq. (6.3) in the limit $\varepsilon_{\Phi Y1} \ll \varepsilon_{Y1}, \varepsilon_{Y2}, \varepsilon_{Y3}$.

It is known [18] that, for $\varepsilon_{Y1}, \varepsilon_{Y2}, \varepsilon_{Y3} > 0$, V_{Σ_Y} is minimized at

$$\Sigma_Y = \sigma\, U\, U^T \qquad ; \qquad \sigma = \sqrt{\frac{\varepsilon_{Y1}}{2\,(\varepsilon_{Y2} + 3\,\varepsilon_{Y3})}}\,, \tag{D.1}$$

where U is an arbitrary 3-by-3 unitary matrix. Let

$$U_\Phi^\dagger\, \Phi\, \Phi^\dagger\, U_\Phi = \begin{pmatrix} u_1^2 & 0 & 0 \\ 0 & u_2^2 & 0 \\ 0 & 0 & u_3^2 \end{pmatrix} \equiv M_d^2 \qquad ; \qquad u_i > 0\,. \tag{D.2}$$

We assume that all u_i's are different. Substituting eqs. (D.1), (D.2) to $V_{\Phi\Sigma_Y}$, it is expressed as

$$V_{\Phi\Sigma_Y} = -\varepsilon_{\Phi Y1}\, \sigma^2\, \mathrm{tr}\left(W^\dagger\, M_d^2\, W\, M_d^2\right) \qquad ; \qquad W = U_\Phi^\dagger\, U\, U^T\, U_\Phi^*\,. \tag{D.3}$$

W is unitary. Define

$$M_d^2 = \mathcal{A}^\alpha\, T^\alpha\,, \qquad W^\dagger\, M_d^2\, W = \mathcal{B}^\alpha\, T^\alpha \qquad ; \qquad \mathcal{A}^\alpha,\, \mathcal{B}^\alpha \in \mathbf{R}\,. \tag{D.4}$$

Then $\mathcal{A}^\alpha \mathcal{A}^\alpha = \mathcal{B}^\alpha \mathcal{B}^\alpha$, since $\mathrm{tr}\left[\left(W^\dagger\, M_d^2\, W\right)^2\right] = \mathrm{tr}\left(M_d^4\right)$. Hence, $V_{\Phi\Sigma_Y} = -\frac{1}{2}\,\varepsilon_{\Phi Y1}\,\sigma^2\,\mathcal{A}^\alpha \mathcal{B}^\alpha$ is minimized when $\mathcal{A}^\alpha = \mathcal{B}^\alpha$. This means $W = U_d$ and $UU^T = U_\Phi U_d U_\Phi^T$, where U_d is an arbitrary diagonal unitary matrix defined in eq. (3.18); it can be absorbed into a redefinition of U_Φ as $U'_\Phi = U_\Phi U_d^{1/2}$.

D.2 Relation between $\overline{Y}$ and Σ_Y

$\overline{Y}$ is in the $({}_{\boldsymbol{nm}}\boldsymbol{C_2}, Q_Y)$ under $\mathrm{SU}(nm) \times \mathrm{U}(1)$, where ${}_{\boldsymbol{nm}}\boldsymbol{C_2}$ stands for the second-rank antisymmetric representation of $\mathrm{SU}(nm)$. $\overline{Y}$ is defined to be unitary. Thus,

$$\overline{Y}^{\xi\eta} = -\overline{Y}^{\eta\xi} \qquad ; \qquad \overline{Y}^{\xi\eta}\, \overline{Y}^{\zeta\eta^*} = \delta^{\xi\zeta}\,. \tag{D.5}$$

The indices take values $0 \le \xi, \eta, \zeta, \cdots \le nm - 1$.

An orthonormal basis of n-by–m matrices is denoted by $\{\overline{T}^{\xi}\}$ with the normalization condition

$$\mathrm{tr}\left(\overline{T}^{\xi\dagger}\overline{T}^{\eta}\right) = \mathrm{tr}\left(\overline{T}^{\xi}\,\overline{T}^{\eta\dagger}\right) = \frac{1}{2}\,\delta^{\xi\eta}\,. \tag{D.6}$$

In particular, the first 9 bases are taken as

$$\overline{T}^{\xi}_{ij} = \begin{cases} T^{\xi}_{ij} & 1 \le i,j \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad (0 \le \xi \le 8)\,. \tag{D.7}$$

We may identify $\Sigma_Y(\mathbf{6}, 1, Q_Y)$ embedded in $\overline{Y}$ as follows.

$$\tilde{Y}_{ik;jl} = \overline{Y}^{\xi\eta}\,\overline{T}^{\xi}_{ij}\,\overline{T}^{\eta}_{kl} \quad \overset{\text{equiv.}}{\Longleftrightarrow} \quad \overline{Y}^{\xi\eta} = 4\,\tilde{Y}_{ik;jl}\,\overline{T}^{\xi\,*}_{ij}\,\overline{T}^{\eta\,*}_{kl}\,, \tag{D.8}$$

$$\begin{pmatrix}\tilde{Y}_{SA}\\ \tilde{Y}_{AS}\end{pmatrix}_{ik;jl} = \frac{1}{4}\left(\tilde{Y}_{ik;jl} \pm \tilde{Y}_{ki;jl} \mp \tilde{Y}_{ik;lj} - \tilde{Y}_{ki;lj}\right) \tag{D.9}$$

$$= \frac{1}{2}\left(\tilde{Y}_{ik;jl} \pm \tilde{Y}_{ki;jl}\right)\,, \tag{D.10}$$

$$(\Sigma_Y)_{ik} = (\tilde{Y}_{SA})_{ik;45} \qquad \text{for } 1 \le i,k \le 3\,. \tag{D.11}$$

D.3 A vacuum of the $\mathrm{SU}(nm) \times \mathrm{U}(1)$-invariant potential

We analyze a vacuum configuration of the $\mathrm{SU}(nm) \times \mathrm{U}(1)$-invariant potential $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\Phi\,\overline{X}\,\overline{Y}}$ given by eq. (6.8). We restrict our analysis to the case $(n,m) = (4,5)$ and consider only $C(p_i, p'_i, q_i, q'_i)$ for $p_i, p'_i \le 1$ and arbitrary q_i, q'_i, while all other $C(p_i, p'_i, q_i, q'_i)$ are set equal to zero. In this restricted parameter space spanned by $\{C(p_i, p'_i, q_i, q'_i)\}$, we examine if the configuration given by eqs. (6.5)–(6.7) can minimize $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\Phi\,\overline{X}\,\overline{Y}}$. We assume that the U(1) charge vanishes,

$$Q_{\mathrm{tot}} \equiv q_i \sum_i Q(z_i(p_i)) - q'_i \sum_i Q(z_i(p'_i)) = 0\,, \tag{D.12}$$

only in the sector for which $\sum_i (q_i + q'_i) > 1$. This is not a strong condition: Except when Q_X and Q_Y satisfy specific relations, this condition is met.

We have checked the following two properties. (I) At each point of the parameter space, the first derivative of $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\Phi\,\overline{X}\,\overline{Y}}$ vanishes at the configuration eqs. (6.5)–(6.7), if

$$\sigma \le \frac{1}{2}\,. \tag{D.13}$$

(II) The configuration eqs. (6.5)–(6.7) minimizes $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\Phi\,\overline{X}\,\overline{Y}}$, if the condition (D.13) is satisfied and at each point in a hypersurface S in the parameter space; the hypersurface S is defined by the condition $C(p_i, p'_i, q_i, q'_i) \ge 0$ if $p_i = p'_i$ and $q_i = q'_i$ for all i, while $C(p_i, p'_i, q_i, q'_i) = 0$ if $p_i \ne p'_i$ or $q_i \ne q'_i$ for any i. These two properties (I)(II) ensure that, of each point in S, there exists a neighborhood, which has a non-zero volume, and in which $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\Phi\,\overline{X}\,\overline{Y}}$ is minimized by the configuration in question. Namely, there exists a finite volume (non-zero measure) in the parameter space (at least) in a neighborhood of S, in which the desired configuration becomes a vacuum.

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The above properties (I)(II) are verified in the following manner. It suffices to show that all $z_i(p_i)$ for $p_i \leq 1$ can be brought to zero simultaneously at the configuration eqs. (6.5)–(6.7) by appropriately adjusting components of $\overline{Y}$ except for Σ_Y. In fact, in this case, $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\Phi\overline{X}\,\overline{Y}}$ as well as its first derivative vanish at any point of the parameter space. Thus, property (I) follows. Since $V^{\mathrm{SU}(nm)\times\mathrm{U}(1)}_{\Phi\overline{X}\,\overline{Y}} \geq 0$ in S, the property (II) follows as well. We have checked numerically that all $z_i(p_i)$ can be brought to zero at the configuration eqs. (6.5)–(6.7) by explicitly constructing the corresponding $\overline{Y}$ for a given value of σ. This turned out to be possible (at least) if the condition (D.13) is met, since there are quite large degrees of freedom in the choice of $\overline{Y}$. (If σ is too large, it conflicts the unitarity condition of $\overline{Y}$.)

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